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COMPUTATION OF A CONTRACTION METRIC FOR A PERIODIC ORBIT USING MESHFREE COLLOCATION

PETER GIESL*

Abstract. Contraction analysis uses a local criterion to prove the long-term behaviour of a dynamical system. We consider a contraction metric, i.e. a Riemannian metric with respect to which the distance between adjacent solutions contracts. If adjacent solutions in all directions perpendicular to the flow are contracted, then there exists a unique periodic orbit, which is exponentially stable.

In this paper we propose a construction method using meshfree collocation to approximately solve a matrix-valued PDE problem. We derive error estimates and show that the approximation is itself a contraction metric if the collocation points are sufficiently dense. We apply the method to several examples.

Keywords. Periodic orbit; contraction metric; matrix-valued Partial Differential Equation; meshfree collocation; error estimates.

AMS subject classifications. 34C25, 65N15, 37C27, 65N35.

1. Introduction.

Ordinary differential equations arise in many applications in biology, physics, and various other areas. The determination of periodic orbits, their stability and basins of attraction are important to analyze systems and to develop models.

We consider a general autonomous ODE of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{f} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. We denote the solution of the initial value problem (1.1) with $\mathbf{x}(0) = \boldsymbol{\xi}$ by $S_t\boldsymbol{\xi} = \mathbf{x}(t)$ and assume that it exists for all $t \geq 0$.

The basin of attraction of a periodic orbit can be determined using a Lyapunov function, however, its definition requires the exact position of the periodic orbit. A contraction metric, on the other hand, can prove the existence, uniqueness and stability of a periodic orbit without knowledge of its position. Moreover, a contraction metric is robust to small perturbations of the system or the metric. This means that a sufficiently good approximation to a certain contraction metric, e.g. using numerical methods, is itself a contraction metric.

A contraction metric is a Riemannian metric such that the distance between adjacent trajectories decreases over time with respect to the Riemannian metric. This type of stability, comparing adjacent solution with each other, is called incremental stability and a contraction metric is a special type of a Finsler-Lyapunov function [4].

A contraction metric for a periodic orbit can be expressed as a matrix-valued function $M \in C^1(\mathbb{R}^n, \mathbb{S}^{n\times n})$, where $\mathbb{S}^{n\times n}$ denotes the symmetric $\mathbb{R}^{n\times n}$ matrices, such that $M(\mathbf{x})$ is positive definite and thus $\langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{x}} = \mathbf{v}^T M(\mathbf{x})\mathbf{w}$ defines a point-dependent scalar product for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. The contraction condition is expressed by $L_M(\mathbf{x}) \leq -\nu < 0$, where L_M is defined in (1.3) below.

We first define for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$

$$V(\mathbf{x}) = D\mathbf{f}(\mathbf{x}) - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T(D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T)}{\|\mathbf{f}(\mathbf{x})\|^2}. \tag{1.2}$$

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Then we define

$$L_M(\mathbf{x}) = \max_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} L_M(\mathbf{x};\mathbf{v}) \tag{1.3}$$
$$L_M(\mathbf{x};\mathbf{v}) = \frac{1}{2}\mathbf{v}^T\Big(M'(\mathbf{x}) + V(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})V(\mathbf{x})\Big)\mathbf{v}\,.$$

Here, $(M'(\mathbf{x}))_{i,j=1,\dots,n} = (\nabla M_{ij}(\mathbf{x}))^T\mathbf{f}(\mathbf{x})$ is the matrix of the orbital derivatives of M_{ij} along solutions of (1.1) and $\|\cdot\| = \|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^n$.

The function $L_M(\mathbf{x};\mathbf{v})$ for $\mathbf{v}$ with $\mathbf{v}^T\mathbf{f}(\mathbf{x}) = 0$ is negative, if the distance between solutions through $\mathbf{x}$ and $\mathbf{x}+\delta\mathbf{v}$ for small $\delta > 0$ with respect to the metric $M(\mathbf{x})$ decreases. Let us give a heuristic explanation. To measure the distance, we synchronize the times such that the difference vector between the solutions is perpendicular to the flow. In particular, we define $\theta(t)$ such that $\theta(0) = 0$ and

$$\left(S_{\theta(t)}(\mathbf{x}+\delta\mathbf{v}) - S_t\mathbf{x}\right)^T \mathbf{f}(S_t\mathbf{x}) = 0 \text{ for all } t \geq 0.$$

The implicit function theorem shows that

$$\dot\theta(0) = \frac{\|\mathbf{f}(\mathbf{x})\|^2 - \delta\mathbf{v}^T D\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})}{\mathbf{f}(\mathbf{x}+\delta\mathbf{v})^T\mathbf{f}(\mathbf{x})} \;\approx\; 1 - \delta\frac{\mathbf{v}^T(D\mathbf{f}(\mathbf{x})^T + D\mathbf{f}(\mathbf{x}))\mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2 + \delta\mathbf{v}^T D\mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x})} \tag{1.4}$$

for small $\delta > 0$. Now we consider the squared distance between the trajectories with respect to the Riemannian metric

$$d(t) = \left(S_{\theta(t)}(\mathbf{x}+\delta\mathbf{v}) - S_t\mathbf{x}\right)^T M(S_t\mathbf{x})\left(S_{\theta(t)}(\mathbf{x}+\delta\mathbf{v}) - S_t\mathbf{x}\right)$$

and take the derivative. We obtain, using Taylor expansion,

$$\begin{aligned}
\frac{d}{dt}d(t)\Big|_{t=0} &= \Big(\dot\theta(0)\mathbf{f}(\mathbf{x}+\delta\mathbf{v}) - \mathbf{f}(\mathbf{x})\Big)^T M(\mathbf{x})\delta\mathbf{v} + \delta^2\mathbf{v}^T M'(\mathbf{x})\mathbf{v}\\
&\quad +\delta\mathbf{v}^T M(\mathbf{x})\Big(\dot\theta(0)\mathbf{f}(\mathbf{x}+\delta\mathbf{v}) - \mathbf{f}(\mathbf{x})\Big)\\
&\approx \delta(\dot\theta(0)-1)[\mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{v} + \mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x})]\\
&\quad +\delta^2\dot\theta(0)[(D\mathbf{f}(\mathbf{x})\mathbf{v})^T M(\mathbf{x})\mathbf{v} + \mathbf{v}^T M(\mathbf{x})D\mathbf{f}(\mathbf{x})\mathbf{v}] + \delta^2\mathbf{v}^T M'(\mathbf{x})\mathbf{v}\\
&\approx \delta^2\Bigg[-\frac{\mathbf{v}^T(D\mathbf{f}(\mathbf{x})^T + D\mathbf{f}(\mathbf{x}))\mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2}[\mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{v} + \mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x})]\\
&\quad +(D\mathbf{f}(\mathbf{x})\mathbf{v})^T M(\mathbf{x})\mathbf{v} + \mathbf{v}^T M(\mathbf{x})D\mathbf{f}(\mathbf{x})\mathbf{v} + \mathbf{v}^T M'(\mathbf{x})\mathbf{v}\Bigg] \text{ by (1.4)}\\
&= \delta^2\mathbf{v}^T\left[V(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})V(\mathbf{x}) + M'(\mathbf{x})\right]\mathbf{v} = 2\delta^2 L_M(\mathbf{x};\mathbf{v})\,.
\end{aligned}$$

If $L_M(\mathbf{x};\mathbf{v})$ is bounded by a negative constant $-\nu$, then $d(t)$ is exponentially decreasing.

We now cite the following implication for the existence, uniqueness and stability of a periodic orbit and its basin of attraction, see [7, Theorem 2.1].

THEOREM 1.1. *Let $K \subset \mathbb{R}^n$ be compact, connected and positively invariant set that does not contain an equilibrium of* (1.1)*, i.e. for all $\mathbf{x}\in K$ we have $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$.*

Let $M \in C^1(\mathbb{R}^n, \mathbb{S}^{n\times n})$ such that $M(\mathbf{x})$ is positive definite for all $\mathbf{x}\in\mathbb{R}^n$. Moreover, assume that $L_M(\mathbf{x}) \leq -\nu < 0$ holds for all $\mathbf{x}\in K$, see (1.3)*.*

Then there exists a unique periodic orbit $\Omega \subset K$, Ω is exponentially stable and the largest real part of all non-trivial Floquet exponents is at most $-\nu$. Moreover, K is a subset of the basin of attraction $A(\Omega)$.

A matrix-valued function M satisfying the assumptions of Theorem 1.1 is called contraction metric. It provides information about the basin of attraction of a periodic orbit, without requiring any information about its existence or location.

Contraction metrics for periodic orbits have been studied by Borg [1] with the Euclidean metric and Stenström [21] with a general Riemannian metric. For further results on contraction metrics see [13, 14, 15, 16].

The converse question, namely the existence of a contraction metric defined in the basin of attraction of an exponentially stable periodic orbit, has been studied in [5]. In [7], a contraction metric was characterized as the unique solution of a matrix-valued PDE.

Computational methods for contraction methods have been proposed in [8] for periodic orbits in time-periodic systems, where the contraction metric was a continuous piecewise affine (CPA) function and the contraction conditions were transformed into constraints of a semidefinite optimization problem. This is similar to the construction of a Lyapunov function, which, however, can be solved using linear optimization, where much larger optimization problems can be tackled. The phase space is triangulated and the number of constraints becomes very large. While the method includes error estimates, which guarantee that the feasibility of the semidefinite optimization problem implies the rigorous determination of a contraction metric, the method is computationally demanding.

In [17, Theorem 3] a contraction metric for periodic orbits was constructed using Linear Matrix Inequalities and SOS (sum of squares). This method is applicable to polynomial systems, and can be generalized. The computational demand depends on the required degree of the polynomial for the construction of the contraction metric. Due to the fact that not all positive polynomials can be written as a sum of squared polynomials, the method is not always guaranteed to succeed.

In this paper we propose a method to construct a contraction metric by approximately solving the matrix-valued PDE from [7] using meshfree collocation. Meshfree collocation is a powerful method to solve interpolation and PDE problems, it works in any dimension and does not require a triangulation of the phase space. Instead, the PDE is required to hold at a set of scattered collocation points, the approximation is computed as the solution of a system of linear equations and it is the norm-minimal interpolant in a reproducing kernel Hilbert space, in our case a Sobolev space. The size of the linear system corresponds to the number of collocation points. The method is particularly suitable for refinement due to the scattered collocation points.

It has been shown in [7] that the unique solution of the PDE problem is a contraction metric. In this paper, we approximate the solution of the PDE problem by $\widetilde{M}$ via meshfree collocation and prove that the approximation is also a contraction metric, using error estimates.

Let us give an overview over the paper: in Section 2 we introduce the PDE problem and cite an existence and uniqueness result. In Section 3 we show that if $\widetilde{M}$ is an approximate solution of the PDE problem, then $\widetilde{M}$ is a contraction metric, i.e. $\widetilde{M}$ is positive definite and $L_{\widetilde{M}}$ is negative definite. In Section 4 we discuss meshfree collocation and prove error estimates. Section 5 applies the method to several examples. The appendix provides explicit formulas for the calculations and explains how to check the conditions that $\widetilde{M}$ is positive definite and $L_{\widetilde{M}}$ is negative.

2. PDE characterizing a contraction metric. In [7] the existence and uniqueness of a solution of a linear matrix-valued PDE has been shown, see Theorem 2.1. Moreover, the solution of this problem is a contraction metric.

To introduce the PDE problem, we define for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ the linear differential operator L, acting on $M \in C^1(\mathbb{R}^n, \mathbb{S}^{n\times n})$ by

$$LM(\mathbf{x}) := M'(\mathbf{x}) + V(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})V(\mathbf{x}), \tag{2.1}$$

where V was defined in (1.2). Moreover, we define the projection $P_{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ onto the $(n-1)$-dimensional space perpendicular to $\mathbf{f}(\mathbf{x})$, i.e. $P_{\mathbf{x}}^2 = P_{\mathbf{x}}$, $P_{\mathbf{x}}\mathbf{f}(\mathbf{x}) = \mathbf{0}$ and $P_{\mathbf{x}}\mathbf{v} = \mathbf{v}$ if $\mathbf{v}^T\mathbf{f}(\mathbf{x}) = 0$, by

$$P_{\mathbf{x}} := I - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T}{\|\mathbf{f}(\mathbf{x})\|^2}. \tag{2.2}$$

The following theorem is from [7, Theorem 3.1, Theorem 4.2].

THEOREM 2.1. *Let Ω be an exponentially stable periodic orbit of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^s(\mathbb{R}^n, \mathbb{R}^n)$, where $s \geq 2$, with basin of attraction $A(\Omega)$. Fix $\mathbf{x}_0 \in A(\Omega)$ and $c_0 \in \mathbb{R}^+$. Let $B \in C^{s-1}(A(\Omega), \mathbb{S}^{n\times n})$ be such that $B(\mathbf{x})$ is positive definite for all $\mathbf{x} \in A(\Omega)$ and define $C \in C^{s-1}(A(\Omega), \mathbb{S}^{n\times n})$ by (see (2.2))*

$$C(\mathbf{x}) = P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}}.$$

Then there exists a unique solution $M \in C^{s-1}(A(\Omega), \mathbb{S}^{n\times n})$ of the linear matrix-valued PDE (see (2.1))

$$LM(\mathbf{x}) = -C(\mathbf{x}) \text{ for all } \mathbf{x} \in A(\Omega) \tag{2.3}$$

$$\text{satisfying } \mathbf{f}(\mathbf{x}_0)^T M(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0) = c_0\|\mathbf{f}(\mathbf{x}_0)\|^4. \tag{2.4}$$

The solution $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in A(\Omega)$ and it is of the form

$$M(\mathbf{x}) = \int_0^\infty \Phi(t,0;\mathbf{x})^T C(S_t\mathbf{x})\Phi(t,0;\mathbf{x})\,dt + c_0\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T,$$

where $\Phi(t,0;\mathbf{x})$ denotes the principal fundamental matrix solution of $\dot{\boldsymbol{\phi}}(t) = D(S_t\mathbf{x})\boldsymbol{\phi}(t)$ with $\Phi(0,0;\mathbf{x}) = I$.

Note that since $L_M(\mathbf{x};\mathbf{v}) = \frac{1}{2}\mathbf{v}^T LM(\mathbf{x})\mathbf{v}$, see (1.3), a function M satisfying (2.3) gives $L_M(\mathbf{x};\mathbf{v}) = -\frac{1}{2}\mathbf{v}^T P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}}\mathbf{v}$, and thus $L_M(\mathbf{x}) = -\frac{1}{2}\min_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} \mathbf{v}^T B(\mathbf{x})\mathbf{v}$, which can be bounded by a negative constant $-\nu$ for all $\mathbf{x}$ within a compact set $K \subset A(\Omega)$. Moreover, M, satisfying (2.3) and (2.4) is positive definite and, hence, is a contraction metric.

In this paper we seek to approximate the solution of (2.3) and (2.4) by $\widetilde{M}$, satisfying $L\widetilde{M}(\mathbf{x}) = -\widetilde{C}(\mathbf{x})$ by meshfree collocation. Note that we cannot assume that $\widetilde{C}(\mathbf{x})$ is of the form $P_{\mathbf{x}}^T \widetilde{B}(\mathbf{x}) P_{\mathbf{x}}$. We will, however, assume that C and $\widetilde{C}$ are close to each other, as well as their first derivatives. We will show that then the approximation $\widetilde{M}$ is a contraction metric. This proves a constructive converse theorem.

3. Approximation of PDE is a contraction metric. In this section we will show that an approximate solution of (2.3) and (2.4), which is sufficiently close, is a contraction metric, in particular, that it is positively invariant. Sufficiently close is expressed by the fact that the difference between LM and $L\widetilde{M}$, see (3.1), as well as their first derivatives, is sufficiently small, see (3.3) and (3.4). Later, we will show error estimates which ensure that (3.3) and (3.4) are satisfied for $\widetilde{M}$ being an approximation using meshfree collocation. For the following theorem we denote $\gamma^+(K) = \bigcup_{t\geq 0} S_t K$.

THEOREM 3.1. *Let the assumptions of Theorem 2.1 hold. Let $K \subset A(\Omega)$ be a compact set with $\Omega \subset \overset{\circ}{K}$ and let $\mathbf{x}_0 \in K$ as well as $c_0 \in \mathbb{R}^+$.*

Then there is an $\epsilon > 0$ such that for all $\widetilde{M}, \widetilde{C} \in C^1(\overline{\gamma^+(K)}, \mathbb{S}^{n\times n})$ satisfying

$$L\widetilde{M}(\mathbf{x}) = -\widetilde{C}(\mathbf{x}) \text{ for all } \mathbf{x} \in \overline{\gamma^+(K)} \tag{3.1}$$

$$\mathbf{f}(\mathbf{x}_0)^T \widetilde{M}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0) = c_0\|\mathbf{f}(\mathbf{x}_0)\|^4 \tag{3.2}$$

$$\|C(\mathbf{x}) - \widetilde{C}(\mathbf{x})\| \leq \epsilon \text{ for all } \mathbf{x} \in \overline{\gamma^+(K)} \tag{3.3}$$

$$\left\| \frac{d}{dx_i}\left(C(\mathbf{x}) - \widetilde{C}(\mathbf{x})\right)\right\| \leq \epsilon \text{ for all } \mathbf{x} \in \overline{\gamma^+(K)} \text{ and } i = 1, \ldots, n \tag{3.4}$$

we have that $\widetilde{M}(\mathbf{x})$ is positive definite for all $\mathbf{x} \in K$. Moreover, there is a constant $\tilde{\nu} > 0$ such that

$$L_{\widetilde{M}}(\mathbf{x}) \leq -\tilde{\nu}$$

holds for all $\mathbf{x} \in \overline{\gamma^+(K)}$, where L_M was defined in (1.3).

REMARK 3.2. *Note that for a compact and positively invariant set we have $\overline{\gamma^+(K)} = K$.*

Proof. Step I
Denote by $T > 0$ the (minimal) period of the periodic orbit Ω and by $-\nu < 0$ the maximal real part of all its non-trivial Floquet exponents. Define $\mathbf{f}_0(\mathbf{x}) = \frac{\mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2}$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$. Set

$$F_0 := \min_{\mathbf{x}\in\overline{\gamma^+(K)}} \|\mathbf{f}(\mathbf{x})\| > 0, \text{ then}$$

$$\max_{\mathbf{x}\in\overline{\gamma^+(K)}} \|\mathbf{f}_0(\mathbf{x})\| = \max_{\mathbf{x}\in\overline{\gamma^+(K)}} \frac{1}{\|\mathbf{f}(\mathbf{x})\|} = \frac{1}{\min_{\mathbf{x}\in\overline{\gamma^+(K)}}\|\mathbf{f}(\mathbf{x})\|} = \frac{1}{F_0}.$$

$$\text{Define } F := \max_{\mathbf{x}\in\overline{\gamma^+(K)}} \|\mathbf{f}(\mathbf{x})\|,$$

$$F_1 := \max_{i=1,\ldots,n} \max_{\mathbf{x}\in\overline{\gamma^+(K)}} \left\|\frac{\partial}{\partial x_i}\mathbf{f}_0(\mathbf{x})\right\|.$$

Choose $\epsilon_0 = \frac{1}{2}\min(\nu, 1)$. We use [7, Lemma 3.3], which proves the existence of a compact, positively invariant neighborhood U of Ω with $\Omega \subset U^\circ \subset U \subset A(\Omega)$ and a map $\pi \in C^{s-1}(U, \Omega)$ with $\pi(\mathbf{x}) = \mathbf{x}$ if and only if $\mathbf{x} \in \Omega$. For a fixed $\mathbf{x} \in U$ there is a bijective C^{s-1} map $\theta_{\mathbf{x}}\colon [0,\infty) \to [0,\infty)$ with inverse $t_{\mathbf{x}} = \theta_{\mathbf{x}}^{-1} \in C^{s-1}([0,\infty),[0,\infty))$ such that $\theta_{\mathbf{x}}(0) = 0$ and

$$\pi(S_t\mathbf{x}) = S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x})$$

for all $t \in [0,\infty)$. Moreover, $\dot{\theta}_{\mathbf{x}}(t) \in [1-\epsilon_0, 1+\epsilon_0]$ for all $t \geq 0$ and $\dot{t}_{\mathbf{x}}(\theta) \in [1-\epsilon_0, 1+\epsilon_0]$ for all $\theta \geq 0$.

Finally, there is a constant $C > 0$ such that

$$|\dot{\theta}_{\mathbf{x}}(t) - 1| \leq Ce^{-\mu_0 t} \text{ for all } t \geq 0 \tag{3.5}$$

$$\|S_{t_{\mathbf{x}}(\theta)}\mathbf{x} - S_\theta\pi(\mathbf{x})\| \leq Ce^{-\mu_0\theta}\|\mathbf{x} - \pi(\mathbf{x})\| \text{ for all } \theta \geq 0 \tag{3.6}$$

and all $\mathbf{x} \in U$, where $\mu_0 = \nu - \epsilon_0 > 0$; for (3.5) see the proof of [6, Corollary 3.6]. We define $c_U := \max_{\mathbf{x}\in U}\|\mathbf{x} - \pi(\mathbf{x})\|$. Since $K \subset A(\Omega)$ is compact, there is a $T_0 > 0$ such that $S_tK \subset U$ for all $t \geq T_0$.

By [7, Lemma 3.4], with similar arguments as in [7, Lemma 3.5] to extend it to the compact set K, there exists $\kappa > 0$ such that for all $\mathbf{x} \in K$ we have

$$\|P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})\| \leq C_0 e^{-\kappa t} \tag{3.7}$$

for all $t \geq 0$, where $\Phi(t, 0; \mathbf{x})$ denotes the principal fundamental matrix solution of the first variational equation $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)$ with $\Phi(0, 0; \mathbf{x}) = I$.

There are $\Lambda, \lambda > 0$ such that $\mathbf{v}^T B(\mathbf{x})\mathbf{v} \geq \lambda\|\mathbf{v}\|^2$ and $\mathbf{v}^T \widetilde{M}(\mathbf{x})\mathbf{v} \leq \Lambda\|\mathbf{v}\|^2$ hold for all $\mathbf{x} \in \overline{\gamma^+(K)}$ and all $\mathbf{v} \in \mathbb{R}^n$, since B is positive definite and continuous, and $\widetilde{M}$ is continuous on the compact set $\overline{\gamma^+(K)}$.

For $\mathbf{x}$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ define

$$A(\mathbf{x}) = D\mathbf{f}(\mathbf{x}) - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T[D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T]}{\|\mathbf{f}(\mathbf{x})\|^2}. \tag{3.8}$$

We denote by $\mu(A) = \lim_{h\to 0^+} \frac{\|I+hA\|-1}{h}$ the logarithmic norm of a matrix $A \in \mathbb{R}^{n\times n}$, which can also be negative. For the matrix norm $\|\cdot\|$ induced by the vector norm $\|\cdot\| = \|\cdot\|_2$, i.e. $\|A\| = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$, we have that $\mu(A)$ is the largest eigenvalue of $\frac{1}{2}(A^T + A)$. Since the eigenvalues vary continuously with the matrix elements, we can define

$$\alpha_0 := \max_{\mathbf{y}\in\overline{\gamma^+(K)}} \mu(-A(\mathbf{y})). \tag{3.9}$$

We will see in Step III that $\alpha_0 > 0$.

Define the positive constants

$$c_1 := \min\left(\frac{\lambda}{2\alpha_0}, c_0 F^2\right) \tag{3.10}$$

$$c_2 := \frac{T_0}{F_0^2} + \sqrt{n}\frac{2F_1F_0 + 1}{F_0^2}\frac{C}{\mu_0}c_U(1+\epsilon_0) + \frac{T}{F_0^2} + \frac{C}{F_0^2\mu_0(1-\epsilon_0)} \tag{3.11}$$

$$c_3 := \frac{C_0^2}{2\kappa} + 2\frac{C_0F}{F_0\kappa} + 2c_2F^2 \tag{3.12}$$

$$\epsilon := \min\left(\frac{\lambda}{2}, \frac{c_1}{2c_3}\right). \tag{3.13}$$

Step II

We first show $L_{\widetilde{M}}(\mathbf{x}) \leq -\tilde{\nu} := -\frac{\lambda}{4\Lambda}$ for all $\mathbf{x} \in \overline{\gamma^+(K)}$. We have for all $\mathbf{x} \in \overline{\gamma^+(K)}$, since $\mathbf{v}^T\mathbf{f}(\mathbf{x}) = 0$ implies $P_{\mathbf{x}}\mathbf{v} = \mathbf{v}$

$$\begin{aligned}
2L_{\widetilde{M}}(\mathbf{x}) \leq & \max_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T\widetilde{M}(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} \mathbf{v}^T LM(\mathbf{x})\mathbf{v} \\
& + \max_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T\widetilde{M}(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} \mathbf{v}^T[L\widetilde{M}(\mathbf{x}) - LM(\mathbf{x})]\mathbf{v} \\
\leq & -\min_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T\widetilde{M}(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} \mathbf{v}^T P_{\mathbf{x}}^T B(\mathbf{x})P_{\mathbf{x}}\mathbf{v} \\
& + \max_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T\widetilde{M}(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} \mathbf{v}^T[\widetilde{C}(\mathbf{x}) - C(\mathbf{x})]\mathbf{v} \\
= & -\min_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T\widetilde{M}(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} \mathbf{v}^T B(\mathbf{x})\mathbf{v} + \max_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T\widetilde{M}(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} \epsilon\|\mathbf{v}\|^2 \\
\leq & (-\lambda + \epsilon) \max_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T\widetilde{M}(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} \|\mathbf{v}\|^2 \\
\leq & -\frac{\lambda}{2\Lambda},
\end{aligned}$$

using (3.13). This shows the statement.

Step III

To show that $\widetilde{M}$ is positive definite, fix $\mathbf{x} \in K$ and $\mathbf{w} \in \mathbb{R}^n$. We write with $c = \mathbf{f}(\mathbf{x})^T\mathbf{w} \in \mathbb{R}$

$$\mathbf{w} = \underbrace{P_{\mathbf{x}}\mathbf{w}}_{=:\mathbf{v}} + c\underbrace{\frac{\mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2}}_{=\mathbf{f}_0(\mathbf{x})} \tag{3.14}$$

such that

$$\|\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \frac{c^2}{\|\mathbf{f}(\mathbf{x})\|^2}. \tag{3.15}$$

Note that $\frac{c^2}{\|\mathbf{f}(\mathbf{x})\|^2} \leq \|\mathbf{w}\|^2$, i.e.

$$|c| \leq \|\mathbf{w}\|F \text{ for all } \mathbf{x} \in \overline{\gamma^+(K)}. \tag{3.16}$$

We have

$$\mathbf{w}^T\widetilde{M}(\mathbf{x})\mathbf{w} \geq \mathbf{w}^T M(\mathbf{x})\mathbf{w} - \left|\mathbf{w}^T[\widetilde{M}(\mathbf{x}) - M(\mathbf{x})]\mathbf{w}\right|. \tag{3.17}$$

Let us estimate the first term in (3.17). Using Theorem 2.1 for the form of $M(\mathbf{x})$ we have

$$\begin{aligned}
\mathbf{w}^T M(\mathbf{x})\mathbf{w} &= \int_0^\infty (P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{w})^T B(S_t\mathbf{x})P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{w}\,dt + c_0\mathbf{w}^T\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T\mathbf{w}\\
&= \int_0^\infty (P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{w})^T B(S_t\mathbf{x})P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{w}\,dt + c_0c^2. \qquad (3.18)
\end{aligned}$$

Note that $\boldsymbol{\phi}(t) = \Phi(t,0;\mathbf{x})\mathbf{w}$ satisfies $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)$. Similar to [7, (3.25)] we have for any solution $\boldsymbol{\phi}(t)$ of $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)$

$$\begin{aligned}
\frac{d}{dt}(P_{S_t\mathbf{x}}\boldsymbol{\phi}(t)) &= \frac{d}{dt}\left(\left(I - \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right)\boldsymbol{\phi}(t)\right)\\
&= -\frac{D\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T + \mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T D\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\boldsymbol{\phi}(t)\\
&\quad + \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T\mathbf{f}(S_t\mathbf{x})^T[D\mathbf{f}(S_t\mathbf{x}) + D\mathbf{f}(S_t\mathbf{x})^T]\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^4}\boldsymbol{\phi}(t)\\
&\quad + \left(I - \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right)D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)\\
&= \left(D\mathbf{f}(S_t\mathbf{x}) - \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T[D\mathbf{f}(S_t\mathbf{x}) + D\mathbf{f}(S_t\mathbf{x})^T]}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right)P_{S_t\mathbf{x}}\boldsymbol{\phi}(t)\\
&= A(S_t\mathbf{x})P_{S_t\mathbf{x}}\boldsymbol{\phi}(t),
\end{aligned}$$

where A was defined in (3.8). By [3, Theorem 3, p. 58] we have

$$\begin{aligned}
\|P_{S_t\mathbf{x}}\boldsymbol{\phi}(t)\| &\geq \|P_{\mathbf{x}}\boldsymbol{\phi}(0)\|\exp\left(-\int_0^t \mu(-A(S_s\mathbf{x}))\,ds\right)\\
&\geq \|P_{\mathbf{x}}\boldsymbol{\phi}(0)\|\exp\left(-\alpha_0 t\right)
\end{aligned}$$

where α_0 was defined in (3.9).

Denoting $\boldsymbol{\phi}(t) = \Phi(t, 0; \mathbf{x})\mathbf{w}$ we have with (3.7)

$$\exp(-\alpha_0 t)\|P_{\mathbf{x}}\mathbf{w}\| \leq \|P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})\mathbf{w}\| \leq C_0 e^{-\kappa t}\|\mathbf{w}\|$$

for any $\mathbf{w} \in \mathbb{R}^n$. Hence, we can conclude that $\alpha_0 > 0$.

Moreover, we have

$$\begin{aligned}
&\int_0^\infty (P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})\mathbf{w})^T B(S_t\mathbf{x}) P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})\mathbf{w}\, dt \\
&\geq \lambda \int_0^\infty \|P_{S_t\mathbf{x}}\boldsymbol{\phi}(t)\|^2\, dt \\
&\geq \lambda \int_0^\infty \|P_{\mathbf{x}}\boldsymbol{\phi}(0)\|^2 \exp(-2\alpha_0 t)\, dt \\
&= \frac{\lambda}{2\alpha_0}\|P_{\mathbf{x}}\mathbf{w}\|^2,
\end{aligned}$$

where λ was defined in Step I.

Altogether, we have with (3.18), (3.10) and (3.15)

$$\begin{aligned}
\mathbf{w}^T M(\mathbf{x})\mathbf{w} &\geq \frac{\lambda}{2\alpha_0}\|\mathbf{v}\|^2 + c_0 c^2 \\
&\geq c_1\|\mathbf{v}\|^2 + \frac{c_1}{F_0^2}c^2 \\
&= c_1\left(\|\mathbf{v}\|^2 + \frac{c^2}{F_0^2}\right) \\
&\geq c_1\|\mathbf{w}\|^2\,. \qquad (3.19)
\end{aligned}$$

Step IV

Now we focus on the second term in (3.17). We have with (3.14)

$$\begin{aligned}
&\left|\mathbf{w}^T[M(\mathbf{x}) - \widetilde{M}(\mathbf{x})]\mathbf{w}\right| \\
&\leq \left|\mathbf{w}^T P_{\mathbf{x}}^T[M(\mathbf{x}) - \widetilde{M}(\mathbf{x})]P_{\mathbf{x}}\mathbf{w}\right| + \left|\mathbf{w}^T P_{\mathbf{x}}^T[M(\mathbf{x}) - \widetilde{M}(\mathbf{x})]c\,\mathbf{f}_0(\mathbf{x})\right| \\
&\quad + \left|c\,\mathbf{f}_0(\mathbf{x})^T[M(\mathbf{x}) - \widetilde{M}(\mathbf{x})]P_{\mathbf{x}}\mathbf{w}\right| + \left|c^2\,\mathbf{f}_0(\mathbf{x})^T[M(\mathbf{x}) - \widetilde{M}(\mathbf{x})]\mathbf{f}_0(\mathbf{x})\right|. \qquad (3.20)
\end{aligned}$$

We will now derive bounds for each of the terms in (3.20). In the following we write $\Phi(t)$ for $\Phi(t, 0; \mathbf{x})$.

Step V: first term in (3.20)

Using Theorem 2.1 for the form of $M(\mathbf{x})$ as well as $\|P_{\mathbf{x}}\mathbf{w}\| \leq \|\mathbf{w}\|$ and

$$P_{\mathbf{x}}^T C(\mathbf{x}) P_{\mathbf{x}} = P_{\mathbf{x}}^T P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}} P_{\mathbf{x}} = P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}} = C(\mathbf{x})$$

we have

$$\left|\mathbf{w}^T P_{\mathbf{x}}^T[M(\mathbf{x}) - \widetilde{M}(\mathbf{x})]P_{\mathbf{x}}\mathbf{w}\right| \leq \left\|P_{\mathbf{x}}^T\widetilde{M}(\mathbf{x})P_{\mathbf{x}} - \int_0^\infty \Phi(s)^T P_{S_s\mathbf{x}}^T C(S_s\mathbf{x}) P_{S_s\mathbf{x}}\Phi(s)\, ds\right\|\, \|\mathbf{w}\|^2.$$

We have by [7, Lemma 4.1, (4.1)]

$$\begin{aligned}
\frac{d}{dt}\left[\Phi(t)^T P_{S_t\mathbf{x}}^T \widetilde{M}(S_t\mathbf{x}) P_{S_t\mathbf{x}}\Phi(t)\right] &= \Phi(t)^T P_{S_t\mathbf{x}}^T L\widetilde{M}(S_t\mathbf{x}) P_{S_t\mathbf{x}}\Phi(t) \\
&= -\Phi(t)^T P_{S_t\mathbf{x}}^T \widetilde{C}(S_t\mathbf{x}) P_{S_t\mathbf{x}}\Phi(t)\,.
\end{aligned}$$

Hence,

$$P_{\mathbf{x}}^T \widetilde{M}(\mathbf{x}) P_{\mathbf{x}} = \Phi(t)^T P_{S_t\mathbf{x}}^T \widetilde{M}(S_t\mathbf{x}) P_{S_t\mathbf{x}} \Phi(t) + \int_0^t \Phi(s)^T P_{S_s\mathbf{x}}^T \widetilde{C}(S_s\mathbf{x}) P_{S_s\mathbf{x}} \Phi(s)\, ds. \tag{3.21}$$

As $t \to \infty$, the first term on the right-hand side vanishes by (3.7). For the second term of (3.21) we have by (3.7)

$$\begin{aligned}
&\left\| \int_0^\infty \Phi(s)^T P_{S_s\mathbf{x}}^T \widetilde{C}(S_s\mathbf{x}) P_{S_s\mathbf{x}} \Phi(s)\, ds - \int_0^\infty \Phi(s)^T P_{S_s\mathbf{x}}^T C(S_s\mathbf{x}) P_{S_s\mathbf{x}} \Phi(s)\, ds \right\| \\
&\qquad \le C_0^2 \epsilon \int_0^\infty e^{-2\kappa s}\, ds \\
&\qquad = \frac{C_0^2}{2\kappa} \epsilon\, .
\end{aligned}$$

Hence, altogether

$$\left| \mathbf{w}^T P_{\mathbf{x}}^T [M(\mathbf{x}) - \widetilde{M}(\mathbf{x})] P_{\mathbf{x}} \mathbf{w} \right| \le \frac{C_0^2}{2\kappa} \epsilon \, \|\mathbf{w}\|^2\, . \tag{3.22}$$

<u>Step VI: second and third terms in (3.20)</u>

Due to the form of $M(\mathbf{x})$, see Theorem 2.1, we have, using $\Phi(t)\mathbf{f}(\mathbf{x}) = \mathbf{f}(S_t\mathbf{x})$ and $P_{S_t\mathbf{x}}\mathbf{f}(S_t\mathbf{x}) = \mathbf{0}$

$$\begin{aligned}
M(\mathbf{x})\mathbf{f}_0(\mathbf{x}) &= \frac{1}{\|\mathbf{f}(\mathbf{x})\|^2} \int_0^\infty \Phi(t)^T P_{S_t\mathbf{x}}^T B(S_t\mathbf{x}) P_{S_t\mathbf{x}} \Phi(t) \mathbf{f}(\mathbf{x})\, dt + c_0 \mathbf{f}(\mathbf{x}) \\
&= \frac{1}{\|\mathbf{f}(\mathbf{x})\|^2} \int_0^\infty \Phi(t)^T P_{S_t\mathbf{x}}^T B(S_t\mathbf{x}) P_{S_t\mathbf{x}} \mathbf{f}(S_t\mathbf{x})\, dt + c_0 \mathbf{f}(\mathbf{x}) \\
&= c_0 \mathbf{f}(\mathbf{x})\, .
\end{aligned}$$

Hence, $\mathbf{f}_0(\mathbf{x})^T M(\mathbf{x}) P_{\mathbf{x}} \mathbf{w} = 0$. This shows

$$\begin{aligned}
\left| c\, \mathbf{f}_0(\mathbf{x})^T [M(\mathbf{x}) - \widetilde{M}(\mathbf{x})] P_{\mathbf{x}} \mathbf{w} \right| &= \left| c\, \mathbf{f}_0(\mathbf{x})^T \widetilde{M}(\mathbf{x}) P_{\mathbf{x}} \mathbf{w} \right| \\
&\le |c|\, \|\mathbf{w}\| \, \left\| \mathbf{f}_0(\mathbf{x})^T \widetilde{M}(\mathbf{x}) P_{\mathbf{x}} \right\|\, .
\end{aligned}$$

We have by [7, Lemma 4.1, (4.4)]

$$\begin{aligned}
\frac{d}{dt}\left[\mathbf{f}_0(S_t\mathbf{x})^T \widetilde{M}(S_t\mathbf{x}) P_{S_t\mathbf{x}} \Phi(t) \right] &= \mathbf{f}_0(S_t\mathbf{x})^T L\widetilde{M}(S_t\mathbf{x}) P_{S_t\mathbf{x}} \Phi(t) \\
&= -\mathbf{f}_0(S_t\mathbf{x})^T \widetilde{C}(S_t\mathbf{x}) P_{S_t\mathbf{x}} \Phi(t) \\
\mathbf{f}_0(\mathbf{x})^T \widetilde{M}(\mathbf{x}_0) P_{\mathbf{x}} &= \mathbf{f}_0(S_t\mathbf{x})^T \widetilde{M}(S_t\mathbf{x}_0) P_{S_t\mathbf{x}} \Phi(t) \\
&\quad + \int_0^t \mathbf{f}_0(S_s\mathbf{x})^T \widetilde{C}(S_s\mathbf{x}) P_{S_s\mathbf{x}} \Phi(s)\, ds.
\end{aligned}$$

As $t \to \infty$, the first term on the right-hand side vanishes by (3.7). For the second term, we have with (3.7) and since $C(S_s\mathbf{x})\mathbf{f}_0(S_s\mathbf{x}) = P_{S_s\mathbf{x}}^T B(S_s\mathbf{x}) P_{S_s\mathbf{x}} \mathbf{f}_0(S_s\mathbf{x}) = \mathbf{0}$

$$\begin{aligned}
\left\| \int_0^\infty \mathbf{f}_0(S_s\mathbf{x})^T \widetilde{C}(S_s\mathbf{x}) P_{S_s\mathbf{x}} \Phi(s)\, ds \right\| &= \left\| \int_0^\infty \mathbf{f}_0(S_s\mathbf{x})^T [\widetilde{C}(S_s\mathbf{x}) - C(S_s\mathbf{x})] P_{S_s\mathbf{x}} \Phi(s)\, ds \right\| \\
&\le \frac{C_0}{F_0} \epsilon \int_0^\infty e^{-\kappa s}\, ds \\
&= \frac{C_0}{F_0 \kappa} \epsilon.
\end{aligned}$$

Hence, altogether by (3.16)

$$\begin{aligned}\left|c\,\mathbf{f}_0(\mathbf{x})^T[M(\mathbf{x})-\widetilde{M}(\mathbf{x})]P_{\mathbf{x}}\mathbf{w}\right| &\leq |c|\,\|\mathbf{w}\|\,\frac{C_0}{F_0\kappa}\epsilon\\ &\leq \|\mathbf{w}\|^2\,\frac{C_0F}{F_0\kappa}\epsilon\,. \end{aligned}\tag{3.23}$$

A similar estimate holds true for the second term in (3.20).

Step VII: last term in (3.20)

We have by [7, Lemma 4.1, (4.2)]

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{f}_0(S_t\mathbf{x})^T\widetilde{M}(S_t\mathbf{x})\mathbf{f}_0(S_t\mathbf{x})\right] &= \mathbf{f}_0(S_t\mathbf{x})^TL\widetilde{M}(S_t\mathbf{x})\mathbf{f}_0(S_t\mathbf{x})\\ &= -\mathbf{f}_0(S_t\mathbf{x})^T\widetilde{C}(S_t\mathbf{x})\mathbf{f}_0(S_t\mathbf{x})\end{aligned}$$

and hence

$$\begin{aligned}&\mathbf{f}_0(S_t\mathbf{x})^T\widetilde{M}(S_t\mathbf{x})\mathbf{f}_0(S_t\mathbf{x})\\ &= \mathbf{f}_0(\mathbf{x})^T\widetilde{M}(\mathbf{x})\mathbf{f}_0(\mathbf{x})-\int_0^t\mathbf{f}_0(S_s\mathbf{x})^T\widetilde{C}(S_s\mathbf{x})\mathbf{f}_0(S_s\mathbf{x})\,ds.\end{aligned}\tag{3.24}$$

We have for $\mathbf{x}\in K$ and $t\geq T_0$, noting that $\mathbf{f}_0(S_s\mathbf{x})^TC(S_s\mathbf{x})\mathbf{f}_0(S_s\mathbf{x})=0$,

$$\begin{aligned}\left\|\int_0^t\mathbf{f}_0(S_s\mathbf{x})^T\widetilde{C}(S_s\mathbf{x})\mathbf{f}_0(S_s\mathbf{x})\,ds\right\| &\leq \int_0^{T_0}\left\|\mathbf{f}_0(S_s\mathbf{x})^T[\widetilde{C}(S_s\mathbf{x})-C(S_s\mathbf{x})]\mathbf{f}_0(S_s\mathbf{x})\right\|\,ds\\ &\quad+\left\|\int_{T_0}^t\mathbf{f}_0(S_s\mathbf{x})^T[\widetilde{C}(S_s\mathbf{x})-C(S_s\mathbf{x})]\mathbf{f}_0(S_s\mathbf{x})\,ds\right\|\\ &\leq \frac{T_0}{F_0^2}\epsilon+\left\|\int_{T_0}^t[g(S_s\mathbf{x})-g(\pi(S_s\mathbf{x}))]\,ds\right\|\\ &\quad+\left\|\int_{T_0}^t g(\pi(S_s\mathbf{x}))\,ds\right\|,\end{aligned}$$

where we define $g(\mathbf{x})=\mathbf{f}_0(\mathbf{x})^T[\widetilde{C}(\mathbf{x})-C(\mathbf{x})]\mathbf{f}_0(\mathbf{x})$.

For any $s\in[T_0,t]$ we have $S_s\mathbf{x}\in U$. Moreover, the straight line between $S_s\mathbf{x}$ and $\pi(S_s\mathbf{x})$ is in U. Thus, since g is C^1, there is a $\theta\in[0,1]$ with

$$g(S_s\mathbf{x})-g(\pi(S_s\mathbf{x}))=\nabla g(\theta S_s\mathbf{x}+(1-\theta)\pi(S_s\mathbf{x}))\cdot(S_s\mathbf{x}-\pi(S_s\mathbf{x})).$$

In particular,

$$\|g(S_s\mathbf{x})-g(\pi(S_s\mathbf{x}))\|\leq\max_{\mathbf{y}\in U}\|\nabla g(\mathbf{y})\|\,\|S_s\mathbf{x}-\pi(S_s\mathbf{x})\|.$$

For ∇g we have

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(\mathbf{f}_0(\mathbf{x})^T[\widetilde{C}(\mathbf{x})-C(\mathbf{x})]\mathbf{f}_0(\mathbf{x})\right) &= \left(\frac{\partial}{\partial x_i}\mathbf{f}_0(\mathbf{x})\right)^T[\widetilde{C}(\mathbf{x})-C(\mathbf{x})]\mathbf{f}_0(\mathbf{x})\\ &\quad+\mathbf{f}_0(\mathbf{x})^T\frac{\partial}{\partial x_i}[\widetilde{C}(\mathbf{x})-C(\mathbf{x})]\mathbf{f}_0(\mathbf{x})\\ &\quad+\mathbf{f}_0(\mathbf{x})^T[\widetilde{C}(\mathbf{x})-C(\mathbf{x})]\left(\frac{\partial}{\partial x_i}\mathbf{f}_0(\mathbf{x})\right)\end{aligned}$$

and thus

$$\max_{\mathbf{y}\in U}\|\nabla g(\mathbf{y})\| \leq \sqrt{n}\epsilon\left(2\frac{F_1}{F_0}+\frac{1}{F_0^2}\right).$$

Denote $\mathbf{p} := S_{T_0}\mathbf{x} \in U$ and $\mathbf{q} := \pi(\mathbf{p}) \in \Omega$. We have $\pi(S_{T_0+s}\mathbf{x}) = \pi(S_s\mathbf{p}) = S_{\theta_{\mathbf{p}}(s)}\pi(\mathbf{p}) \in \Omega$. Since $\mathbf{q} \in \Omega$ is a point on the periodic orbit with period T we have with $\mathbf{f}_0(S_s\mathbf{q})^T C(S_s\mathbf{q})\mathbf{f}_0(S_s\mathbf{q}) = 0$ and by (3.24)

$$\int_0^T g(S_s\mathbf{q})\,ds \;=\; \int_0^T \mathbf{f}_0(S_s\mathbf{q})^T\widetilde{C}(S_s\mathbf{q})\mathbf{f}_0(S_s\mathbf{q})\,ds = 0. \tag{3.25}$$

Altogether, we obtain

$$\begin{aligned}
&\left\|\int_0^t \mathbf{f}_0(S_s\mathbf{x})^T\widetilde{C}(S_s\mathbf{x})\mathbf{f}_0(S_s\mathbf{x})\,ds\right\| \\
&\leq \frac{T_0}{F_0^2}\epsilon + \sqrt{n}\epsilon\frac{2F_1F_0+1}{F_0^2}\int_{T_0}^t \|S_s\mathbf{x}-\pi(S_s\mathbf{x})\|\,ds + \left\|\int_{T_0}^t g(\pi(S_s\mathbf{x}))\,ds\right\| \\
&\leq \frac{T_0}{F_0^2}\epsilon + \sqrt{n}\epsilon\frac{2F_1F_0+1}{F_0^2}\int_0^{t-T_0} \|S_s\mathbf{p}-S_{\theta_{\mathbf{p}}(s)}\pi(\mathbf{p})\|\,ds + \left\|\int_0^{t-T_0} g(\pi(S_s\mathbf{p}))\,ds\right\| \\
&\leq \frac{T_0}{F_0^2}\epsilon + \sqrt{n}\epsilon\frac{2F_1F_0+1}{F_0^2}\int_0^{\theta_{\mathbf{p}}(t-T_0)} \|S_{t_{\mathbf{p}}(\theta)}\mathbf{p}-S_\theta\pi(\mathbf{p})\|\,d\theta(1+\epsilon_0) + \left\|\int_0^{t-T_0} g(S_{\theta_{\mathbf{p}}(s)}\mathbf{q})\,ds\right\| \\
&\leq \frac{T_0}{F_0^2}\epsilon + \sqrt{n}\epsilon\frac{2F_1F_0+1}{F_0^2}C\int_0^{\theta_{\mathbf{p}}(t-T_0)} e^{-\mu_0\theta}\,d\theta\,c_U(1+\epsilon_0) + \left\|\int_0^{\theta_{\mathbf{p}}(t-T_0)} \frac{g(S_\tau\mathbf{q})}{\dot\theta_{\mathbf{p}}(\tau)}\,d\tau\right\| \quad \text{by (3.6)} \\
&\leq \frac{T_0}{F_0^2}\epsilon + \sqrt{n}\epsilon\frac{2F_1F_0+1}{F_0^2}\frac{C}{\mu_0}c_U(1+\epsilon_0) + \left\|\int_0^{\theta_{\mathbf{p}}(t-T_0)} g(S_\tau\mathbf{q})\,d\tau\right\| \\
&\quad + \left\|\int_0^{\theta_{\mathbf{p}}(t-T_0)} g(S_\tau\mathbf{q})\left(\frac{1}{\dot\theta_{\mathbf{p}}(\tau)}-1\right)d\tau\right\|.
\end{aligned}$$

Let us now focus on the last two terms. We have, using (3.25)

$$\left\|\int_0^{\theta_{\mathbf{p}}(t-T_0)} g(S_\tau\mathbf{q})\,d\tau\right\| \leq \max_{t\in[0,T]}\left\|\int_0^t g(S_\tau\mathbf{q})\,d\tau\right\| \leq \frac{T\epsilon}{F_0^2}.$$

Furthermore, using (3.5), we have

$$\begin{aligned}
\left\|\int_0^{\theta_{\mathbf{p}}(t-T_0)} g(S_\tau\mathbf{q})\,\frac{1-\dot\theta_{\mathbf{p}}(\tau)}{\dot\theta_{\mathbf{p}}(\tau)}\,d\tau\right\| &\leq \frac{\epsilon}{F_0^2(1-\epsilon_0)}C\int_0^\infty e^{-\mu_0\tau}\,d\tau \\
&= \frac{\epsilon C}{F_0^2\mu_0(1-\epsilon_0)}.
\end{aligned}$$

Altogether, using (3.24), we obtain

$$\left|\mathbf{f}_0(S_t\mathbf{x})^T\widetilde{M}(S_t\mathbf{x})\mathbf{f}_0(S_t\mathbf{x}) - \mathbf{f}_0(\mathbf{x})^T\widetilde{M}(\mathbf{x})\mathbf{f}_0(\mathbf{x})\right| \leq c_2\epsilon \tag{3.26}$$

for all $t \geq 0$ and all $\mathbf{x} \in K$, see (3.11).

By (3.2) we have for $\mathbf{x} = \mathbf{x}_0$ and all $t \geq 0$

$$\begin{aligned}\left|\mathbf{f}_0(S_t\mathbf{x}_0)^T\widetilde{M}(S_t\mathbf{x}_0)\mathbf{f}_0(S_t\mathbf{x}_0) - \mathbf{f}_0(\mathbf{x}_0)^T\widetilde{M}(\mathbf{x}_0)\mathbf{f}_0(\mathbf{x}_0)\right| &= \left|\mathbf{f}_0(S_t\mathbf{x}_0)^T\widetilde{M}(S_t\mathbf{x}_0)\mathbf{f}_0(S_t\mathbf{x}_0) - c_0\right| \\ &\leq c_2\epsilon\,.\end{aligned}$$

For a fixed point $\mathbf{q} \in \Omega$ there is a sequence $t_n \to \infty$ with $S_{t_n}\mathbf{x}_0 \to \mathbf{q}$ as $n \to \infty$; this shows that

$$\left|\mathbf{f}_0(\mathbf{q})^T\widetilde{M}(\mathbf{q})\mathbf{f}_0(\mathbf{q}) - c_0\right| \leq c_2\epsilon \tag{3.27}$$

holds for all $\mathbf{q} \in \Omega$.

For a fixed point $\mathbf{x} \in K \subset A(\Omega)$ and a fixed $\mathbf{q} \in \Omega$ there is a sequence $t_n \to \infty$ with $S_{t_n}\mathbf{x} \to \mathbf{q}$ as $n \to \infty$ and thus by the form of M as well as (3.26) and (3.27)

$$\begin{aligned}\left|\mathbf{f}_0(\mathbf{x})^T\widetilde{M}(\mathbf{x})\mathbf{f}_0(\mathbf{x}) - \mathbf{f}_0(\mathbf{x})^T M(\mathbf{x})\mathbf{f}_0(\mathbf{x})\right| &= \left|\mathbf{f}_0(\mathbf{x})^T\widetilde{M}(\mathbf{x})\mathbf{f}_0(\mathbf{x}) - c_0\right| \\ &\leq \left|\mathbf{f}_0(\mathbf{x})^T\widetilde{M}(\mathbf{x})\mathbf{f}_0(\mathbf{x}) - \mathbf{f}_0(S_{t_n}\mathbf{x})^T\widetilde{M}(S_{t_n}\mathbf{x})\mathbf{f}_0(S_{t_n}\mathbf{x})\right| \\ &\quad + \left|\mathbf{f}_0(S_{t_n}\mathbf{x})^T\widetilde{M}(S_{t_n}\mathbf{x})\mathbf{f}_0(S_{t_n}\mathbf{x}) - \mathbf{f}_0(\mathbf{q})^T\widetilde{M}(\mathbf{q})\mathbf{f}_0(\mathbf{q})\right| \\ &\quad + \left|\mathbf{f}_0(\mathbf{q})^T\widetilde{M}(\mathbf{q})\mathbf{f}_0(\mathbf{q}) - c_0\right| \\ &\leq 2c_2\epsilon, \end{aligned} \tag{3.28}$$

letting $t_n \to \infty$.

Step VIII

We have now derived the bounds for (3.20) and obtain with (3.22), (3.23) and (3.28)

$$\begin{aligned}\left|\mathbf{w}^T[M(\mathbf{x}) - \widetilde{M}(\mathbf{x})]\mathbf{w}\right| &\leq \frac{C_0^2}{2\kappa}\epsilon\,\|\mathbf{w}\|^2 + 2\|\mathbf{w}\|^2\,\frac{C_0F}{F_0\kappa}\epsilon + 2c_2F^2\epsilon\|\mathbf{w}\|^2 \text{ using (3.16)} \\ &= \epsilon\|\mathbf{w}\|^2c_3, \end{aligned} \tag{3.29}$$

see (3.12).

Hence, we have by (3.17) and (3.19)

$$\mathbf{w}^T\widetilde{M}(\mathbf{x})\mathbf{w} \geq c_1\|\mathbf{w}\|^2 - c_3\epsilon\|\mathbf{w}\|^2 \geq \frac{c_1}{2}\|\mathbf{w}\|^2$$

by (3.13), which shows that $\widetilde{M}(\mathbf{x})$ is positive definite. □

4. Meshfree collocation and error estimates. Meshfree collocation, in particular by radial basis functions, is used to approximate multivariate functions and approximately solve partial differential equations [19, 2, 20]. For a general introduction to meshfree collocation and reproducing kernel Hilbert spaces see [23].

We use meshfree collocation to find an approximation $\widetilde{M}$ to the solution of the matrix-valued PDE (2.3) with condition (2.4). Such a framework has been developed in [12] and we will adapt it to our situation. Moreover, we show error estimates which will ensure that $\widetilde{M}$ satisfies the assumptions of Theorem 3.1, and thus proving a constructive converse theorem.

Denote by $W = \mathbb{S}^{n\times n}$ the space of real-valued symmetric $n \times n$ matrices, which is a separable Hilbert space with inner product

$$\langle \alpha, \beta\rangle_W = \sum_{i,j=1}^{n} \alpha_{ij}\beta_{ij}, \qquad \alpha = (\alpha_{ij}), \beta = (\beta_{ij})$$

and orthonormal basis $\{E^s_{\mu\nu} \in \mathbb{S}^{n\times n} : 1 \leq \mu \leq \nu \leq n\}$. Here, $E^s_{\mu\mu}$ denotes the matrix with value 1 at position (μ, μ) and value zero everywhere else. For $\mu < \nu$, $E^s_{\mu\nu}$ denotes the matrix with value $1/\sqrt{2}$ at positions (μ, ν) and (ν, μ) and value zero everywhere else. We also define $E_{\mu\nu} \in \mathbb{R}^{n\times n}$ to be the matrix with value 1 at position (μ, ν) and value zero everywhere else.

Let $O \subset \mathbb{R}^n$ be a domain with Lipschitz-continuous boundary and consider a mapping $\Phi : O\times O \to \mathcal{L}(W)$, where $\mathcal{L}(W)$ denotes the linear space of all linear and bounded operators $L\colon W \to W$. Φ can be represented by a tensor of order 4, i.e. we let $\Phi = (\Phi_{ijk\ell})$ and define its action on $\alpha \in \mathbb{R}^{n\times n}$ by

$$(\Phi(\mathbf{x}, \mathbf{y})\alpha)_{ij} = \sum_{k,\ell=1}^{n} \Phi(\mathbf{x}, \mathbf{y})_{ijk\ell}\alpha_{k\ell}. \tag{4.1}$$

Let us introduce reproducing kernel Hilbert spaces with values in the Hilbert space W.

Definition 4.1. *The Hilbert space $\mathcal{H}(O; W)$ of functions $g\colon O \to W$ with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}(O;W)}$ is called a* reproducing kernel Hilbert space *if there is a function $\Phi : O \times O \to \mathcal{L}(W)$ with*

1. $\Phi(\cdot, \mathbf{x})\alpha \in \mathcal{H}(O; W)$ for all $\mathbf{x} \in O$ and all $\alpha \in W$.

2. $\langle g(\mathbf{x}), \alpha\rangle_W = \langle g(\cdot), \Phi(\cdot, \mathbf{x})\alpha\rangle_{\mathcal{H}(O;W)}$ for all $g \in \mathcal{H}(O; W)$, all $\mathbf{x} \in O$ and all $\alpha \in W$.

The function Φ is called the reproducing kernel *of $\mathcal{H}(O; W)$.*

Let $\sigma > n/2$ and let $\phi\colon O \times O \to \mathbb{R}$ be a positive definite, reproducing kernel of the Sobolev space $H^\sigma(O; \mathbb{R})$. For example, Wendland's compactly supported radial basis function $\psi_{\ell,k}\colon \mathbb{R}^+_0 \to \mathbb{R}$, see [22], with $\ell = \lfloor \frac{n}{2} \rfloor + k + 1$, $k \in \mathbb{N}$ is a reproducing kernel of $H^\sigma(O; \mathbb{R})$ with $\sigma = k + \frac{n+1}{2}$ with equivalent norm.

By [12, Lemma 3.2, Corollary 3.3], $H^\sigma(O; \mathbb{S}^{n\times n})$ is a reproducing kernel Hilbert space with positive definite reproducing kernel Φ defined by

$$\Phi(\mathbf{x}, \mathbf{y})_{ijk\ell} := \phi(\mathbf{x}, \mathbf{y})\delta_{ik}\delta_{j\ell} \tag{4.2}$$

for $\mathbf{x}, \mathbf{y} \in O$ and $1 \leq i, j, k, \ell \leq n$. Note that $\Phi(\mathbf{x}, \mathbf{y})$ maps $\mathbb{S}^{n\times n}$ to $\mathbb{S}^{n\times n}$.

After having introduced the relevant reproducing kernel Hilbert spaces, let us now focus on the approximation of the PDE problem under consideration. In particular, given $\widetilde{N}$ linearly independent functionals $\lambda_1, \ldots, \lambda_{\widetilde{N}} \in H^\sigma(O; \mathbb{S}^{n\times n})^*$ and the $\widetilde{N}$ values $r_k \in \mathbb{R}$, $k = 1, \ldots, \widetilde{N}$, given by an element $M \in H^\sigma(O; \mathbb{S}^{n\times n})$ through $r_k = \lambda_k(M)$, we seek to determine the optimal recovery of M, defined to be $\widetilde{M} \in H^\sigma(O; \mathbb{S}^{n\times n})$ solving

$$\min\{\|\widetilde{M}\|_{H^\sigma(O;\mathbb{S}^{n\times n})}\colon \widetilde{M} \in H^\sigma(O; \mathbb{S}^{n\times n}) \text{ with } \lambda_i(\widetilde{M}) = r_i, i = 1, \ldots, \widetilde{N}\}.$$

The solution is given by, see [12, Corollary 2.7]

$$\widetilde{M}(\mathbf{x}) = \sum_{k=1}^{\widetilde{N}} \beta_k \sum_{1\leq\mu\leq\nu\leq n} \lambda^{\mathbf{y}}_k(\Phi(\mathbf{y}, \mathbf{x})E^s_{\mu\nu})E^s_{\mu\nu}, \tag{4.3}$$

where the superscript $\mathbf{y}$ denotes the application of the functional with respect to $\mathbf{y}$ and the coefficients $\beta_k \in \mathbb{R}$ are determined by

$$\lambda^{\mathbf{x}}_i\left(\sum_{k=1}^{\widetilde{N}} \beta_k \sum_{1\leq\mu\leq\nu\leq n} \lambda^{\mathbf{y}}_k(\Phi(\mathbf{y}, \mathbf{x})E^s_{\mu\nu})E^s_{\mu\nu}\right) = r_i$$

for $i = 1, \ldots, \widetilde{N}$.

For the approximation of (2.3) and (2.4) we define for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$

$$V(\mathbf{x}) = D\mathbf{f}(\mathbf{x}) - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T(D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T)}{\|\mathbf{f}(\mathbf{x})\|^2} \tag{4.4}$$

so that, see (2.1)

$$LM(\mathbf{x}) = M'(\mathbf{x}) + V(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})V(\mathbf{x}).$$

We set $\sigma = s - 1$ and require $\sigma > n/2 + 1$. This is to ensure that for $M \in H^\sigma(O; \mathbb{S}^{n\times n})$ we have $LM \in H^{\sigma-1}(O; \mathbb{S}^{n\times n})$ with $\sigma - 1 > n/2$. Hence, LM is continuous and the point evaluations in (4.5) are well defined.

We fix the pairwise distinct collocation points $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset O$ as well as the point $\mathbf{x}_0 \in O$ and assume that $O \subset A(\Omega)$. Define the linear functionals $\lambda_k^{(i,j)}, \lambda_0 \colon H^\sigma(O; \mathbb{S}^{n\times n}) \to \mathbb{R}$ for $1 \le i \le j \le n$, $1 \le k \le N$ by

$$\begin{aligned} \lambda_k^{(i,j)}(M) &:= e_i^T LM(\mathbf{x}_k) e_j & (4.5)\\ &=: e_i^T L_k M e_j, & (4.6)\\ \lambda_0(M) &= \mathbf{f}(\mathbf{x}_0)^T M(\mathbf{x}_0) \mathbf{f}(\mathbf{x}_0). & (4.7) \end{aligned}$$

In the next theorem we will show the linear independence of these functionals.

THEOREM 4.2. *Let $O \subset A(\Omega)$ be a domain with Lipschitz boundary. Let $\sigma > n/2 + 1$ and let $\Phi : O \times O \to \mathcal{L}(\mathbb{S}^{n\times n})$ be a reproducing kernel of $H^\sigma(O; \mathbb{S}^{n\times n})$ and let $s = \sigma + 1$. Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subseteq O$ be pairwise distinct points and $\mathbf{x}_0 \in O$ such that $\mathbf{f}(\mathbf{x}_i) \neq \mathbf{0}$ for all $i = 0, \ldots, N$. Let $c_0 \in \mathbb{R}^+$, and let $\lambda_k^{(i,j)}, \lambda_0 \in H^\sigma(O; \mathbb{S}^{n\times n})^*$, $1 \le k \le N$ and $1 \le i \le j \le n$ be defined by (4.5) and (4.7).*

Then these functionals are linearly independent and there is a unique function $\widetilde{M} \in H^\sigma(O; \mathbb{S}^{n\times n})$ solving

$$\min\Big\{ \|\widetilde{M}\|_{H^\sigma(O;\mathbb{S}^{n\times n})} : \lambda_k^{(i,j)}(\widetilde{M}) = -C_{ij}(\mathbf{x}_k), 1 \le i \le j \le n, 1 \le k \le N \\ \text{and } \lambda_0(\widetilde{M}) = c_0 \|\mathbf{f}(\mathbf{x}_0)\|^4 \Big\},$$

where $C(\mathbf{x}) = P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}}$ and $B(\mathbf{x}) = (B_{ij}(\mathbf{x}))_{i,j=1,\ldots,n}$ is a symmetric, positive definite matrix for each $\mathbf{x} \in O$.

It has the form

$$\begin{aligned} \widetilde{M}(\mathbf{x}) = \sum_{k=1}^N \sum_{1\le i\le j\le n} \gamma_k^{(i,j)} \Bigg[&\sum_{\mu=1}^n L_k(\Phi(\cdot,\mathbf{x})_{\cdot,\cdot,\mu,\mu})_{ij} E_{\mu\mu} \\ &+ \frac{1}{2} \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^n [L_k(\Phi(\cdot,\mathbf{x})_{\cdot,\cdot,\mu,\nu})_{ij} + L_k(\Phi(\cdot,\mathbf{x})_{\cdot,\cdot,\nu,\mu})_{ij}] E_{\mu\nu} \Bigg] \\ &+ \gamma_0 \sum_{i,j=1}^n f_i(\mathbf{x}_0) f_j(\mathbf{x}_0) \Bigg[\sum_{\mu=1}^n \Phi(\mathbf{x}_0,\mathbf{x})_{i,j,\mu,\mu} E_{\mu\mu} \\ &+ \frac{1}{2} \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^n [\Phi(\mathbf{x}_0,\mathbf{x})_{i,j,\mu,\nu} + \Phi(\mathbf{x}_0,\mathbf{x})_{i,j,\nu,\mu}] E_{\mu\nu} \Bigg]. \end{aligned} \tag{4.8}$$

where the coefficients $\gamma_k = (\gamma_k^{(i,j)})_{1\le i\le j\le n}$ *and* $\gamma_0 \in \mathbb{R}$ *are determined by* $\lambda_\ell^{(i,j)}(\widetilde{M}) = -C_{ij}(\mathbf{x}_\ell)$ *for* $1 \le i \le j \le n$, $1 \le \ell \le N$ *and* $\lambda_0(\widetilde{M}) = c_0\|\mathbf{f}(\mathbf{x}_0)\|^4$.

If the kernel Φ *is given by* (4.2), *then* $\widetilde{M}$ *is given by*

$$\widetilde{M}(\mathbf{x}) = \sum_{k=1}^{N}\sum_{i,j=1}^{n}\beta_k^{(i,j)}\sum_{\mu,\nu=1}^{n} L_k(\Phi(\cdot,x)_{\cdot,\cdot,\mu,\nu})_{ij}E_{\mu\nu} \\ +\beta_0\phi(\mathbf{x}_0,\mathbf{x})\mathbf{f}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)^T \tag{4.9}$$

where the coefficients $\beta_k = (\beta_k^{(i,j)})_{1\le i,j\le n} \in \mathbb{S}^{n\times n}$ *and* $\beta_0 \in \mathbb{R}$ *are given by* $\beta_0 = \gamma_0$, $\beta_k^{(i,i)} = \gamma_k^{(i,i)}$ *and* $\beta_k^{(i,j)} = \beta_k^{(j,i)} = \frac{1}{2}\gamma_k^{(i,j)}$ *for* $i < j$.

REMARK 4.3. *We will later solve the system of linear equations for* γ *of size* $N\frac{n(n+1)}{2}+1$, *and then determine the* β. *Note that the system for* γ *is of a smaller size, but the form of* $\widetilde{M}$ *in* (4.9) *is beneficial for computations; see also Appendix A.*

Proof. To show the linear independence we assume that

$$\sum_{k=1}^{N}\sum_{1\le i\le j\le n} d_k^{(i,j)}\lambda_k^{(i,j)} + d_0\lambda_0 = 0 \tag{4.10}$$

on $H^\sigma(O;\mathbb{S}^{n\times n})$ with certain coefficients $d_k^{(i,j)}, d_0 \in \mathbb{R}$. We need to show that all $d_k^{(i,j)} = 0$ and $d_0 = 0$.

Let $g \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$ be a nonnegative, compactly supported function with support $B(0,1)$, satisfying $g(\mathbf{x}) = 1$ on $B(0,1/2)$. Fix $1 \le \ell \le N$, as well as $i^*, j^* \in \{1,\dots,n\}$ with $i^* \le j^*$. Since $\mathbf{f}(\mathbf{x}_\ell) \neq \mathbf{0}$, there is $\iota \in \{1,\dots,n\}$ such that $f_\iota(\mathbf{x}_\ell) \neq 0$.

The function

$$g_\ell(\mathbf{x}) = (\mathbf{x}-\mathbf{x}_\ell)_\iota g\left(\frac{\mathbf{x}-\mathbf{x}_\ell}{q}\right),$$

where $q := \min_{\mathbf{x},\mathbf{y}\in X\cup\{\mathbf{x}_0\},\mathbf{x}\neq\mathbf{y}} \|\mathbf{x}-\mathbf{y}\|$ denotes the separation distance of $X\cup\{\mathbf{x}_0\}$, satisfies $g_\ell(\mathbf{x}_k) = 0$ for all $k = 0,\dots,N$. Moreover, $\partial_i g_\ell(\mathbf{x}_k) = 0$ for all $i = 1,\dots,n$ and $\mathbf{x}_k \neq \mathbf{x}_\ell$. Finally, we have $\partial_i g_\ell(\mathbf{x}_\ell) = 0$ for $i \neq \iota$ and $\partial_\iota g_\ell(\mathbf{x}_\ell) = 1$.

Hence, defining the matrix-valued function $G \in H^\sigma(O;\mathbb{S}^{n\times n})$ by $G(\mathbf{x}) = g_\ell(\mathbf{x})E^s_{i^*j^*}$, we have

$$\begin{aligned} 0 &= \sum_{k=1}^{N}\sum_{1\le i\le j\le n} d_k^{(i,j)}\lambda_k^{(i,j)}(G) + d_0\lambda_0(G) \text{ by (4.10)} \\ &= \sum_{k=1}^{N}\sum_{1\le i\le j\le n} d_k^{(i,j)} e_i^T[G'(\mathbf{x}_k) + V(\mathbf{x}_k)^T G(\mathbf{x}_k) + G(\mathbf{x}_k)V(\mathbf{x}_k)]e_j \\ &\quad +d_0\mathbf{f}(\mathbf{x}_0)^T G(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0) \text{ by (4.5) and (4.7)} \end{aligned}$$

We have

$$\mathbf{e}_{i^*}^T E^s_{i^*j^*}\mathbf{e}_{j^*} = \frac{1}{\sqrt{2}} + \left(1-\frac{1}{\sqrt{2}}\right)\delta_{i^*j^*} = \begin{cases} 1 & \text{for } i^* = j^* \\ \frac{1}{\sqrt{2}} & \text{for } i^* \neq j^* \end{cases}.$$

Hence, by definition of G we have

$$
\begin{aligned}
0 &= \sum_{k=1}^{N} d_k^{(i^*,j^*)} \nabla g_\ell(\mathbf{x}_k) \cdot \mathbf{f}(\mathbf{x}_k) \left[\frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right) \delta_{i^*j^*} \right] \\
&= d_\ell^{(i^*,j^*)} f_\iota(\mathbf{x}_\ell) \left[\frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right) \delta_{i^*j^*} \right],
\end{aligned}
$$

which shows $d_\ell^{(i^*,j^*)} = 0$. Hence, we can conclude $d_\ell^{(i^*,j^*)} = 0$ for all $\ell = 1, \ldots, N$ and $i^*, j^* \in \{1, \ldots, n\}$ with $i^* \leq j^*$.

To show that $d_0 = 0$, choose now $G(\mathbf{x}) = I$. Since all other coefficients vanish, we are left with

$$
\begin{aligned}
0 &= d_0 \lambda_0(G) \\
&= d_0 \mathbf{f}(\mathbf{x}_0)^T G(\mathbf{x}_0) \mathbf{f}(\mathbf{x}_0) \\
&= d_0 \|\mathbf{f}(\mathbf{x}_0)\|_2^2
\end{aligned}
$$

which shows $d_0 = 0$. This shows the linear independence.

By (4.3), the minimiser has the form

$$
\begin{aligned}
\widetilde{M}(\mathbf{x}) = \sum_{k=1}^{N} \sum_{1 \leq i \leq j \leq n} \gamma_k^{(i,j)} & \sum_{1 \leq \mu \leq \nu \leq n} \lambda_k^{(i,j)} (\Phi(\cdot, \mathbf{x}) E_{\mu\nu}^s) E_{\mu\nu}^s \\
& + \gamma_0 \sum_{1 \leq \mu \leq \nu \leq n} \lambda_0 (\Phi(\cdot, \mathbf{x}) E_{\mu\nu}^s) E_{\mu\nu}^s,
\end{aligned}
$$

where the coefficients $\gamma_k = (\gamma_k^{(i,j)})_{1 \leq i \leq j \leq n} \in \mathbb{S}^{n \times n}$ and $\gamma_0 \in \mathbb{R}$ are determined by $\lambda_\ell^{(i,j)}(\widetilde{M}) = -C_{ij}(\mathbf{x}_\ell)$ for $1 \leq i \leq j \leq n$, $1 \leq \ell \leq N$ and $\lambda_0(\widetilde{M}) = c_0 \|\mathbf{f}(\mathbf{x}_0)\|^4$.

We will now show (4.8). Indeed, by (4.1) we have

$$
\left(\Phi(\cdot, \mathbf{x}) E_{\mu\nu}^s\right)_{ij} = \sum_{k,\ell=1}^{n} \Phi(\cdot, \mathbf{x})_{ijk\ell} (E_{\mu\nu}^s)_{k\ell}.
$$

For $\mu = \nu$ we have

$$
\begin{aligned}
\lambda_k^{(i,j)} (\Phi(\cdot, \mathbf{x}) E_{\mu\mu}^s) E_{\mu\mu}^s &= L_k (\Phi(\cdot, \mathbf{x})_{\cdot,\cdot,\mu,\mu})_{ij} E_{\mu\mu} \\
\lambda_0 (\Phi(\cdot, \mathbf{x}) E_{\mu\mu}^s) E_{\mu\mu}^s &= \sum_{i,j=1}^{n} f_i(\mathbf{x}_0) f_j(\mathbf{x}_0) \Phi(\mathbf{x}_0, \mathbf{x})_{i,j,\mu,\mu} E_{\mu\mu} \,.
\end{aligned}
$$

For $\mu < \nu$ we have

$$
\begin{aligned}
& \lambda_k^{(i,j)} (\Phi(\cdot, \mathbf{x}) E_{\mu\nu}^s) E_{\mu\nu}^s \\
&= \frac{1}{\sqrt{2}} \left(L_k (\Phi(\cdot, \mathbf{x})_{\cdot,\cdot,\mu,\nu})_{ij} + L_k (\Phi(\cdot, \mathbf{x})_{\cdot,\cdot,\nu,\mu})_{ij} \right) \frac{1}{\sqrt{2}} (E_{\mu\nu} + E_{\nu\mu}) \\
&= \frac{1}{2} \left(L_k (\Phi(\cdot, \mathbf{x})_{\cdot,\cdot,\mu,\nu})_{ij} + L_k (\Phi(\cdot, \mathbf{x})_{\cdot,\cdot,\nu,\mu})_{ij} \right) (E_{\mu\nu} + E_{\nu\mu})
\end{aligned}
$$

and

$$
\begin{aligned}
& \lambda_0 (\Phi(\cdot, \mathbf{x}) E_{\mu\nu}^s) E_{\mu\nu}^s \\
&= \frac{1}{2} \sum_{i,j=1}^{n} f_i(\mathbf{x}_0) f_j(\mathbf{x}_0) \left(\Phi(\mathbf{x}_0, \mathbf{x})_{i,j,\mu,\nu} + \Phi(\mathbf{x}_0, \mathbf{x})_{i,j,\nu,\mu} \right) (E_{\mu\nu} + E_{\nu\mu}).
\end{aligned}
$$

This shows (4.8).

We define the symmetric matrices $\beta_k \in \mathbb{S}^{n\times n}$ by $\beta_k^{(j,i)} = \beta_k^{(i,j)} = \frac{1}{2}\gamma_k^{(i,j)}$ if $i < j$ and $\beta_k^{(i,i)} = \gamma_k^{(i,i)}$. Moreover, let $\beta_0 = \gamma_0$.

To show (4.9), we follow the arguments in the proof of [12, Theorem 5.2] for the first terms. For λ_0 we use $\Phi(\mathbf{x}_0,\mathbf{x})_{i,j,\mu,\nu} = \phi(\mathbf{x}_0,\mathbf{x})\delta_{i\mu}\delta_{j\nu}$. Hence,

$$\begin{aligned}
\sum_{i,j=1}^n f_i(\mathbf{x}_0)f_j(\mathbf{x}_0)\Bigg[\sum_{\mu=1}^n \Phi(\mathbf{x}_0,\mathbf{x})_{i,j,\mu,\mu}E_{\mu\mu} + \frac{1}{2}\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^n [\Phi(\mathbf{x}_0,\mathbf{x})_{i,j,\mu,\nu} + \Phi(\mathbf{x}_0,\mathbf{x})_{i,j,\nu,\mu}]E_{\mu\nu}\Bigg]& \\
= \sum_{\mu=1}^n f_\mu(\mathbf{x}_0)f_\mu(\mathbf{x}_0)\phi(\mathbf{x}_0,\mathbf{x})E_{\mu\mu}& \\
+\frac{1}{2}\phi(\mathbf{x}_0,\mathbf{x})\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^n [f_\mu(\mathbf{x}_0)f_\nu(\mathbf{x}_0) + f_\nu(\mathbf{x}_0)f_\mu(\mathbf{x}_0)]E_{\mu\nu}& \\
= \phi(\mathbf{x}_0,\mathbf{x})\sum_{\mu,\nu=1}^n f_\mu(\mathbf{x}_0)f_\nu(\mathbf{x}_0)E_{\mu\nu}& \\
= \phi(\mathbf{x}_0,\mathbf{x})\mathbf{f}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)^T\,,&
\end{aligned}$$

which shows the theorem. □

We will now establish an error estimate, which depends on the fill distance of the points $X = \{\mathbf{x}_1,\ldots,\mathbf{x}_N\}$ in O defined by $h_{X,O} = \sup_{\mathbf{x}\in O}\min_{\mathbf{x}_j\in X}\|\mathbf{x}-\mathbf{x}_j\|_2$. We thus show a constructive converse theorem, showing that we can construct a contraction metric via meshless collocation if the fill distance is sufficiently small.

THEOREM 4.4. *Let $\mathbf{f} \in C^s(\mathbb{R}^n;\mathbb{R}^n)$, $\mathbb{N} \ni s > n/2+3$ and set $\sigma = s-1$. Let Ω be an exponentially stable periodic orbit of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with basin of attraction $A(\Omega)$.*

Let $B \in C^\sigma(\mathbb{R}^n,\mathbb{S}^{n\times n})$ such that $B(\mathbf{x})$ is a positive definite matrix for all $\mathbf{x}\in\mathbb{R}^n$ and let $C(\mathbf{x}) = P_{\mathbf{x}}^T B(\mathbf{x})P_{\mathbf{x}}$.

Let $M \in C^\sigma(A(\Omega),\mathbb{S}^{n\times n})$ be the solution of (2.3) and (2.4). Let $O \subseteq A(\Omega)$ be a bounded domain with Lipschitz boundary. Finally, let $\widetilde{M}$ be the optimal recovery from Theorem 4.2. Then, we have the error estimates

$$\|LM - L\widetilde{M}\|_{L_\infty(O;\mathbb{S}^{n\times n})} \le Ch_{X,O}^{\sigma-1-n/2}\|M\|_{H^\sigma(O;\mathbb{S}^{n\times n})} \tag{4.11}$$

$$\|\partial_i LM - \partial_i L\widetilde{M}\|_{L_\infty(O;\mathbb{S}^{n\times n})} \le Ch_{X,O}^{\sigma-2-n/2}\|M\|_{H^\sigma(O;\mathbb{S}^{n\times n})} \tag{4.12}$$

for all $X \subseteq O$ and all $i = 1,\ldots,n$ with sufficiently small $h_{X,O}$. By construction we have

$$\mathbf{f}(\mathbf{x}_0)^T\widetilde{M}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0) = c_0\|\mathbf{f}(\mathbf{x}_0)\|^4 \tag{4.13}$$

Let $K \ni \mathbf{x}_0$ be a compact set such that $\overline{\gamma^+(K)} \subset O$. Then $\widetilde{M}$ is a contraction metric in $\overline{\gamma^+(K)}$, i.e. $\widetilde{M}(\mathbf{x})$ is positive definite for all $\mathbf{x}\in K$ and $L_{\widetilde{M}}(\mathbf{x}) \le -\tilde{\nu} < 0$ for all $\mathbf{x} \in \overline{\gamma^+(K)}$, provided $h_{X,O}$ is sufficiently small.

Proof. Note that $LM - L\widetilde{M} \in H^{\sigma-1}(O;\mathbb{S}^{n\times n})$ vanishes on the set X. We can now apply a result from [18], see [11, Theorem 2.5] for references, to each entry of the matrix $LM - L\widetilde{M}$ to obtain

$$\begin{aligned}
\|LM - L\widetilde{M}\|_{L_\infty(O;\mathbb{S}^{n\times n})} &\le Ch_{X,O}^{\sigma-1-n/2}\|L(M-\widetilde{M})\|_{H^{\sigma-1}(O;\mathbb{S}^{n\times n})}\\
&\le Ch_{X,O}^{\sigma-1-n/2}\|M-\widetilde{M}\|_{H^\sigma(O;\mathbb{S}^{n\times n})}\\
&\le Ch_{X,O}^{\sigma-1-n/2}\|M\|_{H^\sigma(O;\mathbb{S}^{n\times n})},
\end{aligned}$$

since L is a differential operator of order 1. Similarly, we have

$$
\begin{aligned}
\|\partial_i(LM - L\widetilde{M})\|_{L_\infty(O;\mathbb{S}^{n\times n})} &\le \|LM - L\widetilde{M}\|_{W^1_\infty(O;\mathbb{S}^{n\times n})}\\
&\le Ch_{X,O}^{\sigma-2-n/2}\|L(M-\widetilde{M})\|_{H^{\sigma-1}(O;\mathbb{S}^{n\times n})}\\
&\le Ch_{X,O}^{\sigma-2-n/2}\|M-\widetilde{M}\|_{H^{\sigma}(O;\mathbb{S}^{n\times n})}\\
&\le Ch_{X,O}^{\sigma-2-n/2}\|M\|_{H^{\sigma}(O;\mathbb{S}^{n\times n})}.
\end{aligned}
$$

We can now use Theorem 3.1 to conclude that $\widetilde{M}$ is a contraction metric in K if $h_{X,O}$ is sufficiently small. □

5. Examples. Meshfree collocation allows for the use of scattered collocation points. The smaller the fill distance, the smaller is the error. However, the smaller the separation distance, the larger is the condition number of the collocation matrix. To balance the two, we have chosen the collocation points on a grid. The optimal choice would be a hexagonal grid, but here we have used a cartesian one.

5.1. Unit circle. As a first example, we consider the following system

$$
\left\{\begin{array}{lcl} \dot{x} &=& x(1-x^2-y^2)-y\\ \dot{y} &=& y(1-x^2-y^2)+x \end{array}\right. \tag{5.1}
$$

where the unit circle is an exponentially stable periodic orbit and the origin is an unstable equilibrium.

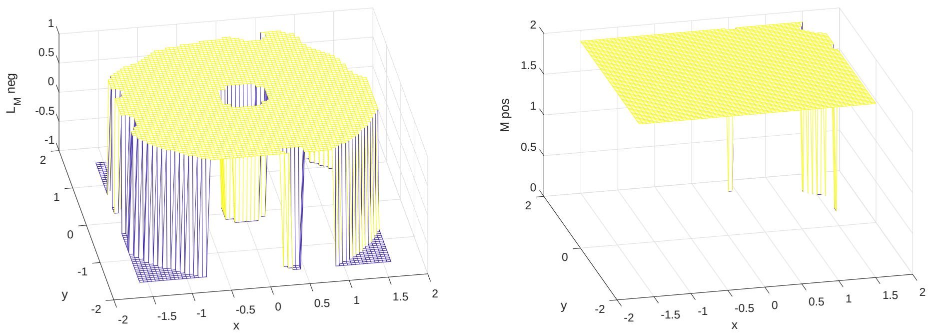

FIG. 5.1. *Left: the area where $L_{\widetilde{M}}(\mathbf{x})$ is negative is the area where the function attains the value 1. Right: the area where $\widetilde{M}(\mathbf{x})$ is positive definite is the area where the function attains the value 2.*

We choose $B(\mathbf{x}) = I$ and the collocation points $X = \frac{1.6}{15}\mathbb{Z}^2 \cap \{(x,y)\in\mathbb{R}^2 \mid 0.25 < \sqrt{x^2+y^2} < 1.5\}$ as well as the point $\mathbf{x}_0 = (1,0)$ with $c_0 = 1$. We use the kernel given by (4.2), where $\phi(\mathbf{x},\mathbf{y}) = \psi_{6,4}(\|\mathbf{x}-\mathbf{y}\|_2)$ is given by the Wendland function $\psi_{6,4}(r) = (1-r)_+^{10}[25+250r+1,050r^2+2,250r^3+2,145r^4]$ and $x_+ = x$ for $x \ge 0$ and $x_+ = 0$ for $x < 0$. The corresponding Sobolev space is $H^{5.5}(O;\mathbb{S}^{2\times 2})$. This results in $N = 600$ collocation points and thus a collocation matrix of size $3N+1 = 1,801$.

The points $\mathbf{x}$ where $L_{\widetilde{M}}(\mathbf{x})$ is negative are those, where the function described in Section B attains 1, see Figure 5.1, left. The points $\mathbf{x}$ where $\widetilde{M}(\mathbf{x})$ is positive definite are those, where the function described in Section B attains 2, see Figure 5.1, right.

Figure 5.2 shows the collocation points, the boundary of the area, where $L_{\widetilde{M}}$ is negative (red), the boundary of the area, where $\widetilde{M}$ is positive definite (blue) and the periodic orbit

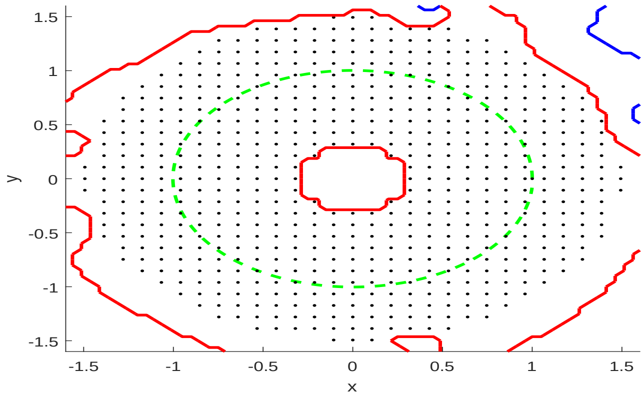

FIG. 5.2. *The collocation points (black), the boundary of the area, where $L_{\widetilde{M}}$ is negative (red), the boundary of the area, where $\widetilde{M}$ is positive definite (blue) and the periodic orbit (green).*

(green). The area, where $L_{\widetilde{M}}$ is negative is the area, where the collocation points are placed, while $\widetilde{M}$ is positive definite in an even larger area.

5.2. Van der Pol. We consider the van der Pol system, given by

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -x + (1-x^2)y \end{cases} \tag{5.2}$$

which has an exponentially stable periodic orbit; the origin is an unstable equilibrium.

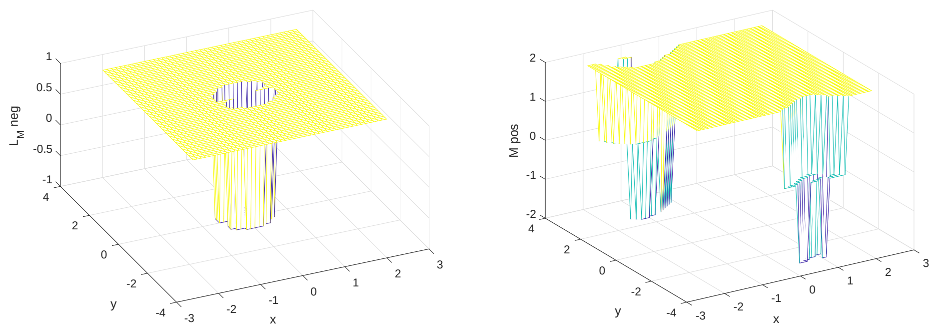

FIG. 5.3. *Left: the area where $L_{\widetilde{M}}(\mathbf{x})$ is negative is the area where the function attains the value 1. Right: the area where $\widetilde{M}(\mathbf{x})$ is positive definite is the area where the function attains the value 2.*

We choose $B(\mathbf{x}) = I$ and the collocation points $X = \left(\frac{2.3}{35}\mathbb{Z} \times \frac{3.1}{45}\mathbb{Z}\right) \cap ([-2.3, 2.3] \times [-3.1, 3.1]) \cap \{(x,y) \in \mathbb{R}^2 \mid 0.8 < \sqrt{x^2+y^2}\}$ as well as the point $\mathbf{x}_0 = (2,0)$ with $c_0 = 1$. We use again the kernel given by the Wendland function $\psi_{6,4}$. This results in $N = 6,022$ collocation points and thus a collocation matrix of size $3N+1 = 18,067$.

The points $\mathbf{x}$ where $L_{\widetilde{M}}(\mathbf{x})$ is negative are those, where the function described in Section B attains 1, see Figure 5.3, left. The points $\mathbf{x}$ where $\widetilde{M}(\mathbf{x})$ is positive definite are those, where the function described in Section B attains 2, see Figure 5.3, right.

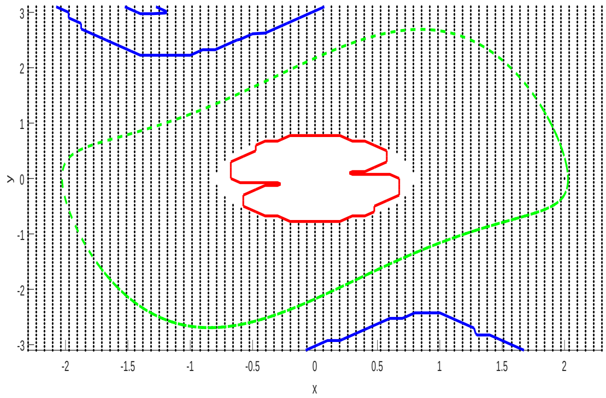

FIG. 5.4. *The collocation points (black), the boundary of the area where $L_{\widetilde{M}}$ is negative (red) and where $\widetilde{M}$ is positive definite (blue), together with the periodic orbit (green).*

Figure 5.4 shows the collocation points (black), the boundary of the area where $L_{\widetilde{M}}$ is negative (red) and where $\widetilde{M}$ is positive definite (blue), together with the periodic orbit (green). The limiting factor is clearly the positive definiteness of $\widetilde{M}$, which does not cover the whole area, where the collocation points are placed. Note that this area is not positively invariant, hence, since we require estimates on $\overline{\gamma^+(K)}$, Theorem 3.1 is only applicable to a smaller set K.

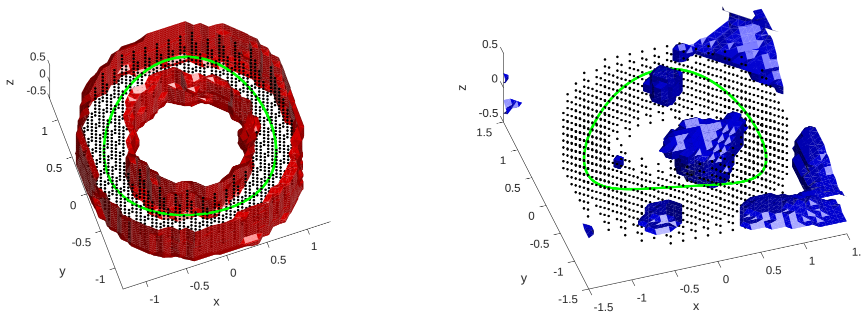

FIG. 5.5. *The collocation points (black) and the periodic orbit (green) together with left: the boundary of the area where $L_{\widetilde{M}}$ is negative (red), right: the boundary of the area where $\widetilde{M}$ is positive definite (blue).*

5.3. Three-dimensional example. We consider the following three-dimensional system

$$\left\{\begin{array}{rcl}\dot{x} &=& x(1-x^2-y^2)-y+0.1yz\\ \dot{y} &=& y(1-x^2-y^2)+x\\ \dot{z} &=& -z+xy\end{array}\right. \tag{5.3}$$

which has an exponentially stable periodic orbit.

We choose $B(\mathbf{x}) = I$ and the collocation points $X = \left(\frac{1.3}{9}\mathbb{Z}^2 \times 0.1\mathbb{Z}\right) \cap \{(x, y, z) \in \mathbb{R}^3 \mid 0.75 < \sqrt{x^2+y^2} < 1.25, |z| < 0.45\}$ as well as the point $\mathbf{x}_0 = (1, 0, 0)$ with $c_0 = 1$. We use

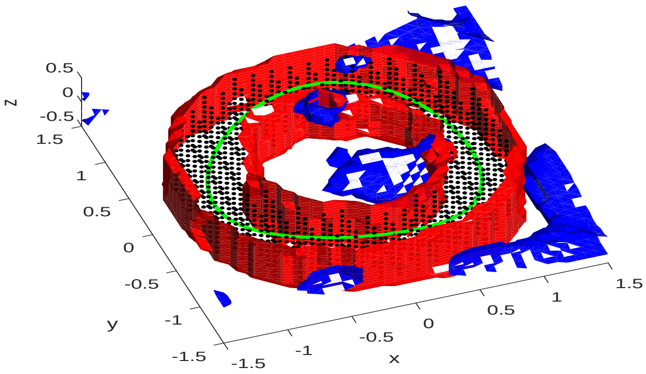

FIG. 5.6. *The collocation points (black), the boundary of the area where $L_{\widetilde{M}}$ is negative (red) and where $\widetilde{M}$ is positive definite (blue), together with the periodic orbit (green).*

again the kernel given by the Wendland function $\psi_{6,4}$, the corresponding Sobolev space is $H^6(O;\mathbb{S}^{3\times 3})$. This results in $N = 1,368$ collocation points and thus a collocation matrix of size $6N + 1 = 8,209$.

Figures 5.5 and 5.6 show the collocation points (black), the boundary of the area where $L_{\widetilde{M}}$ is negative (red) and where $\widetilde{M}$ is positive definite (blue), together with the periodic orbit (green).

6. Conclusions and further work. We have proposed a computationally efficient numerical method to construct a contraction metric for a periodic orbit. A contraction metric is a matrix-valued function, which satisfies a contraction criterion. We consider a certain contraction metric, satisfying a linear, first-order PDE and approximate it using meshfree collocation. We have proved that the approximation, if sufficiently close, is itself a contraction metric. We have obtained error estimates for the meshfree collocation, which show that the approximation is sufficiently close if the collocation points are dense enough.

The method can be further improved by fully exploiting the advantages of meshfree collocation. In particular, one could start with a coarse set of collocation points and refine, where the conditions of a contraction metric are not fulfilled. A further improvement could include a posteriori estimates, that can be obtained by using Taylor-type estimates or by interpolating with a CPA function, similar to [10] and [9] for Lyapunov functions. To determine a positively invariant set, one could first seek to compute a Lyapunov function. In areas where the Lyapunov function does not have negative orbital derivative, such as in a neighborhood of a periodic orbit, we can then employ the method described in this paper. The advantage of this combined method is twofold: on the one hand the computation of a Lyapunov function requires less computational effort, as it is a scalar-valued function. On the other hand, the sublevel set of a Lyapunov function provides us with a compact and positively invariant set.

Appendix A. Explicit formulas for the calculations.

To derive explicit formulas, let us choose a radially symmetric kernel of the form $\phi(\mathbf{x},\mathbf{y}) = \psi_0(\|\mathbf{x}-\mathbf{y}\|_2)$ and denote $\psi_{i+1}(r) = \frac{d\psi_i(r)/dr}{r}$ for $i = 0,1$ and $r > 0$. We assume that ψ_1 and ψ_2 can be continuously extended up to $r = 0$; this is, e.g. the case for sufficiently smooth Wendland functions. We use the kernel Φ of the form (4.2), hence

$$\Phi(\cdot,\mathbf{x})_{ij\mu\nu} = \psi_0(\|\cdot-\mathbf{x}\|_2)\delta_{i\mu}\delta_{j\nu}. \tag{A.1}$$

Thus, for the linear operators L_k, see (4.5), we have

$$\begin{aligned}
(L_k(M))_{ij} &= \sum_{p=1}^n V_{pi}(\mathbf{x}_k)M_{pj}(\mathbf{x}_k) + \sum_{p=1}^n M_{ip}(\mathbf{x}_k)V_{pj}(\mathbf{x}_k)\\
&\quad + \sum_{p=1}^n \partial_p M_{ij}(\mathbf{x}_k) f_p(\mathbf{x}_k)\\
(L_k(\Phi(\cdot,\mathbf{x}))_{\cdot,\cdot,\mu,\nu})_{ij} &= \sum_{p=1}^n \psi_0(\|\mathbf{x}_k-\mathbf{x}\|_2)V_{pi}(\mathbf{x}_k)\delta_{p\mu}\delta_{j\nu}\\
&\quad + \sum_{p=1}^n \psi_0(\|\mathbf{x}_k-\mathbf{x}\|_2)\delta_{i\mu}\delta_{p\nu}V_{pj}(\mathbf{x}_k)\\
&\quad + \sum_{p=1}^n \psi_1(\|\mathbf{x}_k-\mathbf{x}\|_2)(\mathbf{x}_k-\mathbf{x})_p f_p(\mathbf{x}_k)\delta_{i\mu}\delta_{j\nu}\\
&= \psi_0(\|\mathbf{x}_k-\mathbf{x}\|_2)V_{\mu i}(\mathbf{x}_k)\delta_{j\nu} + \psi_0(\|\mathbf{x}_k-\mathbf{x}\|_2)\delta_{i\mu}V_{\nu j}(\mathbf{x}_k)\\
&\quad + \psi_1(\|\mathbf{x}_k-\mathbf{x}\|_2)\langle \mathbf{x}_k-\mathbf{x}, \mathbf{f}(\mathbf{x}_k)\rangle\delta_{i\mu}\delta_{j\nu},
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^n$.

Now we can compute $\widetilde{M}(\mathbf{x})$, using (4.9) of Theorem 4.2. We have

$$\begin{aligned}
\widetilde{M}(\mathbf{x}) &= \sum_{k=1}^N \sum_{i,j=1}^n \beta_k^{(i,j)} \sum_{\mu,\nu=1}^n (L_k(\Phi(\cdot,\mathbf{x}))_{\cdot,\cdot,\mu,\nu})_{ij} E_{\mu\nu}\\
&\quad + \beta_0\psi_0(\|\mathbf{x}_0-\mathbf{x}\|_2)\mathbf{f}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)^T\\
&= \sum_{k=1}^N \Bigg[\sum_{i,\mu,\nu=1}^n \beta_k^{(i,\nu)}\psi_0(\|\mathbf{x}_k-\mathbf{x}\|_2)V_{\mu i}(\mathbf{x}_k)E_{\mu\nu}\\
&\quad + \sum_{j,\mu,\nu=1}^n \beta_k^{(\mu,j)}\psi_0(\|\mathbf{x}_k-\mathbf{x}\|_2)V_{\nu j}(\mathbf{x}_k)E_{\mu\nu}\\
&\quad + \sum_{\mu,\nu=1}^n \beta_k^{(\mu,\nu)}\psi_1(\|\mathbf{x}_k-\mathbf{x}\|_2)\langle \mathbf{x}_k-\mathbf{x}, \mathbf{f}(\mathbf{x}_k)\rangle E_{\mu\nu}\Bigg]\\
&\quad + \beta_0\psi_0(\|\mathbf{x}_0-\mathbf{x}\|_2)\mathbf{f}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)^T\\
&= \sum_{k=1}^N \Bigg[\psi_0(\|\mathbf{x}_k-\mathbf{x}\|_2)\left[V(\mathbf{x}_k)\beta_k + \beta_k V(\mathbf{x}_k)^T\right]\\
&\quad + \psi_1(\|\mathbf{x}_k-\mathbf{x}\|_2)\langle \mathbf{x}_k-\mathbf{x}, \mathbf{f}(\mathbf{x}_k)\rangle\beta_k\Bigg]\\
&\quad + \beta_0\psi_0(\|\mathbf{x}_0-\mathbf{x}\|_2)\mathbf{f}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)^T.
\end{aligned} \tag{A.2}$$

Hence,

$$
\begin{aligned}
L\widetilde{M}(\mathbf{x}) = &\sum_{k=1}^{N} \psi_0(\|\mathbf{x}_k - \mathbf{x}\|_2)\big[V(\mathbf{x})^T V(\mathbf{x}_k)\beta_k + V(\mathbf{x})^T \beta_k V(\mathbf{x}_k)^T \\
&+V(\mathbf{x}_k)\beta_k V(\mathbf{x}) + \beta_k V(\mathbf{x}_k)^T V(\mathbf{x})\big] \\
&+\sum_{k=1}^{N} \psi_1(\|\mathbf{x}_k - \mathbf{x}\|_2)\langle \mathbf{x}_k - \mathbf{x}, \mathbf{f}(\mathbf{x}_k)\rangle \left[V(\mathbf{x})^T \beta_k + \beta_k V(\mathbf{x})\right] \\
&+\sum_{k=1}^{N} \psi_1(\|\mathbf{x}_k - \mathbf{x}\|_2)\langle \mathbf{x} - \mathbf{x}_k, \mathbf{f}(\mathbf{x})\rangle \left[V(\mathbf{x}_k)\beta_k + \beta_k V(\mathbf{x}_k)^T\right] \\
&-\sum_{k=1}^{N} \psi_1(\|\mathbf{x}_k - \mathbf{x}\|_2)\langle \mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}_k)\rangle \beta_k \\
&+\sum_{k=1}^{N} \psi_2(\|\mathbf{x}_k - \mathbf{x}\|_2)\langle \mathbf{x}_k - \mathbf{x}, \mathbf{f}(\mathbf{x}_k)\rangle \langle \mathbf{x} - \mathbf{x}_k, \mathbf{f}(\mathbf{x})\rangle \beta_k \\
&+\beta_0 \psi_0(\|\mathbf{x}_0 - \mathbf{x}\|_2)[V(\mathbf{x})^T \mathbf{f}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)^T + \mathbf{f}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)^T V(\mathbf{x})] \\
&+\beta_0 \psi_1(\|\mathbf{x}_0 - \mathbf{x}\|_2)\langle \mathbf{x} - \mathbf{x}_0, \mathbf{f}(\mathbf{x})\rangle \mathbf{f}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)^T. \qquad \text{(A.3)}
\end{aligned}
$$

Observe that $L\widetilde{M}(\mathbf{x})$ is a symmetric matrix if all β_k, $k = 1, \ldots, N$ are symmetric.

After establishing the formulas for $\widetilde{M}$ and $L\widetilde{M}$, let us now consider the linear system for the coefficients γ and β, respectively.

Let us first calculate the coefficients $b_{(\ell,i,j),(k,\mu,\nu)}$, $b_{0,(k,\mu,\nu)}$, $b_{(\ell,i,j),0}$ and $b_{0,0}$ for $1 \leq k, \ell \leq N$, $1 \leq i, j, \mu, \nu \leq n$ such that

$$
\mathbf{f}(\mathbf{x}_0)^T \widetilde{M}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0) = \sum_{k=1}^{N} \sum_{\mu,\nu=1}^{n} b_{0,(k,\mu,\nu)} \beta_k^{(\mu,\nu)} + b_{0,0}\beta_0. \qquad \text{(A.4)}
$$

$$
L\widetilde{M}(\mathbf{x}_\ell)_{i,j} = \sum_{k=1}^{N} \sum_{\mu,\nu=1}^{n} b_{(\ell,i,j),(k,\mu,\nu)} \beta_k^{(\mu,\nu)} + b_{(\ell,i,j),0}\beta_0. \qquad \text{(A.5)}
$$

We have by (A.2)

$$
\begin{aligned}
\mathbf{f}(\mathbf{x}_0)^T \widetilde{M}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0) = &\sum_{k=1}^{N} \Bigg[\psi_0(\|\mathbf{x}_k - \mathbf{x}_0\|_2)\mathbf{f}(\mathbf{x}_0)^T \left[V(\mathbf{x}_k)\beta_k + \beta_k V(\mathbf{x}_k)^T\right] \mathbf{f}(\mathbf{x}_0) \\
&+\psi_1(\|\mathbf{x}_k - \mathbf{x}_0\|_2)\langle \mathbf{x}_k - \mathbf{x}_0, \mathbf{f}(\mathbf{x}_k)\rangle \mathbf{f}(\mathbf{x}_0)^T \beta_k \mathbf{f}(\mathbf{x}_0)\Bigg] \\
&+\beta_0 \psi_0(0)\|\mathbf{f}(\mathbf{x}_0)\|^4
\end{aligned}
$$

and thus

$$
\begin{aligned}
b_{0,(k,\mu,\nu)} &= \psi_0(\|\mathbf{x}_k - \mathbf{x}_0\|_2)\Bigg[\sum_{p=1}^{n} V_{p\mu}(\mathbf{x}_k) f_p(\mathbf{x}_0) f_\nu(\mathbf{x}_0) + \sum_{p=1}^{n} V_{p\nu}(\mathbf{x}_k) f_p(\mathbf{x}_0) f_\mu(\mathbf{x}_0)\Bigg] \\
&\quad +\psi_1(\|\mathbf{x}_k - \mathbf{x}_0\|_2)\langle \mathbf{x}_k - \mathbf{x}_0, \mathbf{f}(\mathbf{x}_k)\rangle f_\mu(\mathbf{x}_0) f_\nu(\mathbf{x}_0) \qquad \text{(A.6)} \\
b_{0,0} &= \psi_0(0)\|\mathbf{f}(\mathbf{x}_0)\|^4. \qquad \text{(A.7)}
\end{aligned}
$$

By (A.3) we have

$$
\begin{aligned}
b_{(\ell,i,j),(k,\mu,\nu)} = \psi_0(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\Bigg[& \sum_{p=1}^{n} V_{pi}(\mathbf{x}_\ell)V_{p\mu}(\mathbf{x}_k)\delta_{\nu j} + V_{\mu i}(\mathbf{x}_\ell)V_{j\nu}(\mathbf{x}_k) \\
& + V_{i\mu}(\mathbf{x}_k)V_{\nu j}(\mathbf{x}_\ell) + \delta_{i\mu}\sum_{p=1}^{n} V_{p\nu}(\mathbf{x}_k)V_{pj}(\mathbf{x}_\ell)\Bigg] \\
& + \psi_1(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\langle \mathbf{x}_k - \mathbf{x}_\ell, \mathbf{f}(\mathbf{x}_k)\rangle \left[V_{\mu i}(\mathbf{x}_\ell)\delta_{\nu j} + \delta_{i\mu}V_{\nu j}(\mathbf{x}_\ell)\right] \\
& + \psi_1(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\langle \mathbf{x}_\ell - \mathbf{x}_k, \mathbf{f}(\mathbf{x}_\ell)\rangle \left[V_{i\mu}(\mathbf{x}_k)\delta_{\nu j} + \delta_{i\mu}V_{j\nu}(\mathbf{x}_k)\right] \\
& - \psi_1(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\langle \mathbf{f}(\mathbf{x}_\ell), \mathbf{f}(\mathbf{x}_k)\rangle \delta_{i\mu}\delta_{j\nu} \\
& + \psi_2(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\langle \mathbf{x}_k - \mathbf{x}_\ell, \mathbf{f}(\mathbf{x}_k)\rangle\langle \mathbf{x}_\ell - \mathbf{x}_k, \mathbf{f}(\mathbf{x}_\ell)\rangle \delta_{i\mu}\delta_{j\nu}
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
\text{and } b_{(\ell,i,j),0} = \psi_0(\|\mathbf{x}_0 - \mathbf{x}_\ell\|_2)\Bigg[& \sum_{p=1}^{n} V_{pi}(\mathbf{x}_\ell)f_p(\mathbf{x}_0)f_j(\mathbf{x}_0) + \sum_{p=1}^{n} V_{pj}(\mathbf{x}_\ell)f_p(\mathbf{x}_0)f_i(\mathbf{x}_0)\Big] \\
& + \psi_1(\|\mathbf{x}_0 - \mathbf{x}_\ell\|_2)\langle \mathbf{x}_\ell - \mathbf{x}_0, \mathbf{f}(\mathbf{x}_\ell)\rangle f_i(\mathbf{x}_0)f_j(\mathbf{x}_0).
\end{aligned} \tag{A.9}
$$

It is now easy to see that

$$b_{(\ell,i,j),(k,\mu,\nu)} = b_{(\ell,j,i),(k,\nu,\mu)}, \tag{A.10}$$

$$b_{(\ell,i,j),(k,\mu,\nu)} = b_{(k,\mu,\nu),(\ell,i,j)}, \tag{A.11}$$

$$b_{0,(\ell,i,j)} = b_{0,(\ell,j,i)}, \tag{A.12}$$

$$b_{(\ell,i,j),0} = b_{0,(\ell,i,j)}. \tag{A.13}$$

We now compute the $\gamma_k^{(\mu,\nu)}$, which are defined by $\gamma_0 = \beta_0$, $\gamma_k^{(\mu,\mu)} = \beta_k^{(\mu,\mu)}$ and $\frac{1}{2}\gamma_k^{(\mu,\nu)} = \beta_k^{(\mu,\nu)} = \beta_k^{(\nu,\mu)}$ for $\mu < \nu$. They solve the (smaller) linear system

$$
\begin{aligned}
\sum_{k=1}^{N}\sum_{1\le\mu\le\nu\le n} c_{(\ell,i,j),(k,\mu,\nu)}\gamma_k^{(\mu,\nu)} + c_{(\ell,i,j),0}\gamma_0 &= L\widetilde{M}(x_\ell)_{i,j} \\
&= \lambda_\ell^{(i,j)}(\widetilde{M}) \\
&= -C_{ij}(\mathbf{x}_\ell)
\end{aligned} \tag{A.14}
$$

$$\sum_{k=1}^{N}\sum_{1\le\mu\le\nu\le n} c_{0,(k,\mu,\nu)}\gamma_k^{(\mu,\nu)} + c_{0,0}\gamma_0 = c_0\|\mathbf{f}(\mathbf{x}_0)\|^4 \tag{A.15}$$

for $1 \le \ell \le N$, $1 \le i \le j \le n$. The coefficients $c_{\cdot,\cdot}$ form a symmetric (see below) matrix of size $N\frac{n(n+1)}{2} + 1$.

Let us express the $c_{(\ell,i,j),(k,\mu,\nu)}$ in terms of the previously calculated $b_{(\ell,i,j),(k,\mu,\nu)}$.

Noting that

$$
\begin{aligned}
\sum_{k=1}^{N}\sum_{\mu,\nu=1}^{n} b_{(\ell,i,j),(k,\mu,\nu)}\beta_k^{(\mu,\nu)} + b_{(\ell,i,j),0}\beta_0 = & \sum_{k=1}^{N}\sum_{1\le\mu\le\nu\le n} c_{(\ell,i,j),(k,\mu,\nu)}\gamma_k^{(\mu,\nu)} \\
& + c_{(\ell,i,j),0}\gamma_0
\end{aligned} \tag{A.16}
$$

$$
\begin{aligned}
\sum_{k=1}^{N}\sum_{\mu,\nu=1}^{n} b_{0,(k,\mu,\nu)}\beta_k^{(\mu,\nu)} + b_{0,0}\beta_0 = & \sum_{k=1}^{N}\sum_{1\le\mu\le\nu\le n} c_{0,(k,\mu,\nu)}\beta_k^{(\mu,\nu)} \\
& + c_{0,0}\gamma_0
\end{aligned} \tag{A.17}
$$

as well as the definition of $\gamma_k^{(\mu,\mu)}$ and γ_0, we have from the first equation for all (ℓ, i, j)

$$b_{(\ell,i,j),0} = c_{(\ell,i,j),0} \tag{A.18}$$

$$\begin{aligned}
\sum_{k=1}^{N}\sum_{\mu,\nu=1}^{n} b_{(\ell,i,j),(k,\mu,\nu)}\beta_k^{(\mu,\nu)} &= \sum_{k=1}^{N}\sum_{\mu=1}^{n} b_{(\ell,i,j),(k,\mu,\mu)}\beta_k^{(\mu,\mu)} \\
&\quad + \sum_{k=1}^{N}\sum_{1\le\mu<\nu\le n} (b_{(\ell,i,j),(k,\mu,\nu)}\beta_k^{(\mu,\nu)} + b_{(\ell,i,j),(k,\nu,\mu)}\beta_k^{(\nu,\mu)}) \\
&= \sum_{k=1}^{N}\sum_{\mu=1}^{n} b_{(\ell,i,j),(k,\mu,\mu)}\gamma_k^{(\mu,\mu)} \\
&\quad + \sum_{k=1}^{N}\sum_{1\le\mu<\nu\le n} \frac{1}{2}(b_{(\ell,i,j),(k,\mu,\nu)} + b_{(\ell,i,j),(k,\nu,\mu)})\gamma_k^{(\mu,\nu)}. \qquad \text{(A.19)}
\end{aligned}$$

Comparing (A.19) to (A.16) gives, using (A.10)

$$\begin{aligned}
c_{(\ell,i,i),(k,\mu,\mu)} &= b_{(\ell,i,i),(k,\mu,\mu)} \\
c_{(\ell,i,i),(k,\mu,\nu)} &= \frac{1}{2}\left(b_{(\ell,i,i),(k,\mu,\nu)} + b_{(\ell,i,i),(k,\nu,\mu)}\right) = b_{(\ell,i,i),(k,\mu,\nu)} \\
c_{(\ell,i,j),(k,\mu,\mu)} &= b_{(\ell,i,j),(k,\mu,\mu)} = \frac{1}{2}\left(b_{(\ell,i,j),(k,\mu,\mu)} + b_{(\ell,j,i),(k,\mu,\mu)}\right) \\
c_{(\ell,i,j),(k,\mu,\nu)} &= \frac{1}{4}\left(b_{(\ell,i,j),(k,\mu,\nu)} + b_{(\ell,j,i),(k,\nu,\mu)} + b_{(\ell,i,j),(k,\nu,\mu)} + b_{(\ell,j,i),(k,\mu,\nu)}\right) \\
&= \frac{1}{2}\left(b_{(\ell,i,j),(k,\mu,\nu)} + b_{(\ell,i,j),(k,\nu,\mu)}\right) \qquad \text{(A.20)}
\end{aligned}$$

where we assume $\mu < \nu$ and $i < j$.

From(A.17) we have

$$c_{0,0} = b_{0,0} \tag{A.21}$$

$$\begin{aligned}
\sum_{k=1}^{N}\sum_{\mu,\nu=1}^{n} b_{0,(k,\mu,\nu)}\beta_k^{(\mu,\nu)} &= \sum_{k=1}^{N}\sum_{\mu=1}^{n} b_{0,(k,\mu,\mu)}\beta_k^{(\mu,\mu)} \\
&\quad + \sum_{k=1}^{N}\sum_{1\le\mu<\nu\le n} (b_{0,(k,\mu,\nu)}\beta_k^{(\mu,\nu)} + b_{0,(k,\nu,\mu)}\beta_k^{(\nu,\mu)}) \\
&= \sum_{k=1}^{N}\sum_{\mu=1}^{n} b_{0,(k,\mu,\mu)}\gamma_k^{(\mu,\mu)} \\
&\quad + \sum_{k=1}^{N}\sum_{1\le\mu<\nu\le n} \frac{1}{2}(b_{0,(k,\mu,\nu)} + b_{0,(k,\nu,\mu)})\gamma_k^{(\mu,\nu)}. \qquad \text{(A.22)}
\end{aligned}$$

Comparing (A.22) to (A.17) gives, using (A.12)

$$c_{0,(k,\mu,\nu)} = \frac{1}{2}\left(b_{0,(k,\mu,\nu)} + b_{0,(k,\nu,\mu)}\right) \; = \; b_{0,(k,\mu,\nu)}, \tag{A.23}$$

where we assume $\mu < \nu$. The matrix $c_{\cdot,\cdot}$ is symmetric due to (A.11) and (A.13).

Summarising, for the computations we calculate the coefficients $b_{\cdot,\cdot}$ using (A.6), (A.7), (A.8) and (A.9), and then the symmetric matrix $c_{\cdot,\cdot}$ using (A.20), (A.21) and (A.23). Then we determine $\gamma_k^{(\mu,\nu)}$ and γ_0 by solving (A.14) and (A.15) and compute $\beta_k \in \mathbb{S}^{n\times n}$ from γ_k; recall that $\beta_k^{(j,i)} = \beta_k^{(i,j)} = \frac{1}{2}\gamma_k^{(i,j)}$ if $i<j$ and $\beta_k^{(i,i)} = \gamma_k^{(i,i)}$ as well as $\beta_0 = \gamma_0$. $\widetilde{M}(\mathbf{x})$ and $L\widetilde{M}(\mathbf{x})$ are then given by (A.2) and (A.3).

Appendix B. Computation of the condition.

Once the approximation $\widetilde{M}$ is calculated, we seek to show that $\widetilde{M}(\mathbf{x})$ is positive definite and $L_{\widetilde{M}}(\mathbf{x})$ is negative.

To check that $\widetilde{M}(\mathbf{x})$ is positive definite, we calculate the characteristic polynomial of $-\widetilde{M}(\mathbf{x})$:

$$\chi_{-\widetilde{M}(\mathbf{x})}(\lambda) = a_0 + a_1\lambda + \ldots + a_{n-1}\lambda^{n-1} + \lambda^n.$$

Now we can apply the Liénard-Chipart criterion to the coefficients to ensure that $-\widetilde{M}(\mathbf{x})$ is negative definite.

To check that $L_{\widetilde{M}}(\mathbf{x})$ is negative, we need to show that $L\widetilde{M}(\mathbf{x}) = V^T\widetilde{M}(\mathbf{x}) + \widetilde{M}(\mathbf{x})V(\mathbf{x}) + \widetilde{M}'(\mathbf{x})$ where $V(\mathbf{x}) = D\mathbf{f}(\mathbf{x}) - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T(D\mathbf{f}(\mathbf{x})+D\mathbf{f}(\mathbf{x})^T)}{\|\mathbf{f}(\mathbf{x})\|^2}$ satisfies $\mathbf{v}^T L\widetilde{M}(\mathbf{x})\mathbf{v} < 0$ for all $\mathbf{v}$ with $\|\mathbf{v}\| = 1$ and $\mathbf{v} \perp \mathbf{f}(\mathbf{x})$. We consider the symmetric real-valued matrix $P_{\mathbf{x}}^T L\widetilde{M}(\mathbf{x})P_{\mathbf{x}}$. This matrix has one eigenvalue 0 with eigenvector $\mathbf{f}(\mathbf{x})$ and for the negativity we need to show that all other eigenvalues are negative. This follows from the fact that there exists an orthonormal basis of eigenvectors.

We calculate the characteristic polynomial of $P_{\mathbf{x}}^T L\widetilde{M}(\mathbf{x})P_{\mathbf{x}}$:

$$\chi_{P_{\mathbf{x}}^T L\widetilde{M}(\mathbf{x})P_{\mathbf{x}}}(\lambda) = b_0 + b_1\lambda + \ldots + b_{n-1}\lambda^{n-1} + \lambda^n.$$

Since 0 is an eigenvalue, we have $b_0 = (-1)^n \det(P_{\mathbf{x}}^T L\widetilde{M}(\mathbf{x})P_{\mathbf{x}}) = 0$ and we can write the characteristic polynomial as

$$\chi_{P_{\mathbf{x}}^T L\widetilde{M}(\mathbf{x})P_{\mathbf{x}}}(\lambda) = \lambda(b_1 + b_2\lambda + \ldots + b_{n-1}\lambda^{n-2} + \lambda^{n-1}).$$

We now apply the Liénard-Chipart criterion to the coefficients of

$$b_1 + b_2\lambda + \ldots + b_{n-1}\lambda^{n-2} + \lambda^{n-1}.$$

In two-dimensional systems, we have $a_0 = \det(-\widetilde{M}(\mathbf{x})) = \det(\widetilde{M}(\mathbf{x}))$ and $a_1 = -\operatorname{trace}(-\widetilde{M}(\mathbf{x})) = \operatorname{trace}(\widetilde{M}(\mathbf{x}))$ and we require $a_0, a_1 > 0$; similarly $b_1 = -\operatorname{trace}(P_{\mathbf{x}}^T L\widetilde{M}(\mathbf{x})P_{\mathbf{x}})$ and we require $b_1 > 0$. In the examples, we calculate the function $\operatorname{sign}(a_0(\mathbf{x})) + \operatorname{sign}(a_1(\mathbf{x}))$ and determine the area where it has the value 2 and similarly where $\operatorname{sign}(b_1(\mathbf{x}))$ has the value 1.

In three-dimensional systems we have $a_0 = -\det(-\widetilde{M}(\mathbf{x})) = \det(\widetilde{M}(\mathbf{x}))$, $a_1 = \sum_{i=1}^3 \det(-\widetilde{M}(\mathbf{x}))_i = \sum_{i=1}^3 \det(\widetilde{M}(\mathbf{x}))_i$, where A_i denotes the 2×2 matrix obtained by deleting the i-th row and i-th column of A, as well as $a_2 = -\operatorname{trace}(-\widetilde{M}(\mathbf{x})) = \operatorname{trace}(\widetilde{M}(\mathbf{x}))$. We require $a_2, a_0 > 0$ as well as $a_2a_1 - a_0 > 0$.

Moreover, we have $b_1 = \sum_{i=1}^3 \det(P_{\mathbf{x}}^T LM(\mathbf{x})P_{\mathbf{x}})_i$ as well as $b_2 = -\operatorname{trace}(P_{\mathbf{x}}^T LM(\mathbf{x})P_{\mathbf{x}})$ and require $b_2, b_1 > 0$.

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